Some computations of stable twisted homology for mapping class groups

Arthur Soulié

June 16, 2020

Abstract

In this paper, we deal with stable homology computations with twisted coefficients for mapping class groups of surfaces and of 3-manifolds, automorphism groups of free groups with boundaries and automorphism groups of certain right-angled Artin groups. On the one hand, the computations are led using semidirect product structures arising naturally from these groups. On the other hand, we compute the stable homology with twisted coefficients by FI-modules. This notably uses a decomposition result of the stable homology with twisted coefficients for pre-braided monoidal categories proved in this paper.

Introduction

Computing the homology of a group is a fundamental question and can be a very difficult task. For example, a complete understanding of all the homology groups of mapping class groups of surfaces and 3-manifolds remains out of reach at present time: this is an active research topic (see [19, 22, 27] for constant coefficients and [18, 25] for twisted coefficients).

In [28], Randal-Williams and Wahl prove homological stability with twisted coefficients for some families of groups, including mapping class groups of surfaces and 3-manifolds. They consider a set of groups $\{G_n\}_{n\in\mathbb{N}}$ such that there exist a canonical injections $G_n \hookrightarrow G_{n+1}$. Let $\mathcal{G}$ be the groupoid with objects indexed by natural numbers, with the groups $\{G_n\}_{n\in\mathbb{N}}$ as automorphism groups and with no morphisms between distinct objects. We consider Quillen's bracket construction on $\mathcal{G}$ (see [17, p.219]), denoted by $\mathfrak{U}\mathcal{G}$, and $\mathbf{Ab}$ the category of abelian groups. Randal-Williams and Wahl show that for particular kinds of functors $F : \mathfrak{U}\mathcal{G} \to \mathbf{Ab}$ (namely coefficients systems of finite degree, see [28, Section 4]), then the canonical induced maps

$$H_*\left(G_n, F(n)\right) \to H_*\left(G_{n+1}, F(n+1)\right)$$

are isomorphisms for $N(*, d) \leq n$ with some $N(*, d) \in \mathbb{N}$ depending on $*$ and d. The value of the homology for $n \geq N(*, d)$ is called the stable homology of the family of groups $\{G_n\}_{n\in\mathbb{N}}$ and denoted by $H_*(G_\infty, F_\infty)$.

In this paper, we are interested in explicit computations of the stable homology with twisted coefficients for mapping class groups. On the one hand, using semidirect product structures naturally arising from mapping class groups, on the strength of Lyndon-Hochschild-Serre spectral sequence and of certain stability results, we prove:

Theorem A **(Theorems 2.17 and 2.24).** *We have:*

1. *We denote by $\Gamma_{g,1}$ the isotopy classes of diffeomorphisms restricting to the identity on the boundary component of a compact connected orientable surface with one boundary component and genus $g \geq 0$. Then, for m, n and q natural numbers such that $2n \geq 3q + m$, there is an isomorphism:*

$$H_q\left(\Gamma_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right) \cong \bigoplus_{\left\lfloor \frac{q-1}{2} \right\rfloor \geq k \geq 0} H_{q-(2k+1)}\left(\Gamma_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m-1}\right).$$

Hence, we recover the results of [18] and [25].

This work was partially supported by the ANR Project *ChroK*, `ANR-16-CE40-0003` and by a JSPS Postdoctoral Fellowship Short Term.

2010 *Mathematics Subject Classification*: 20J05, 20J06, 20F38, 18D10.

Keywords: Stable homology, Mapping class groups, Functor categories, Polynomial functors.

2. *We consider the graph $\mathscr{G}_{n,k}^{s}$ defined as the wedge of $n \in \mathbb{N}$ circles with $k \in \mathbb{N}$ distinguished circles joined by arcs to the basepoint p_0 and $s-1 \in \mathbb{N}$ extra basepoints joined by new edges to p_0. We denote by $A_{n,k}^{s}$ the group of path-components space of homotopy equivalences of the graph $\mathscr{G}_{n,k}^{s}$ (we refer the reader to Section 2.3.2 for a complete introduction to these groups). In particular it is the quotient of the mapping class group of some 3-manifold (see Remark 2.21). For $s \geq 2$ and $q \geq 1$ be natural numbers and $F : \mathfrak{gr} \to \mathbf{Ab}$ a reduced polynomial functor where $\mathfrak{gr}$ denotes the category of finitely generated free groups. There is an isomorphism:*

$$H_q\left(A_{\infty,0}^{s}, F_\infty\right) = 0.$$

Moreover, $H_q\left(A_{n,k}^{s}, \mathbb{Q}\right) = 0$ for all natural numbers $n \geq 3q+3$ and $k \geq 0$. We thus recover the results of [23] for holomorphs of free groups.

On the other hand, we deal with stable homology computations for mapping class groups with twisted coefficients factoring through some finite groups. Let $(\Sigma, \sqcup, \emptyset)$ be the symmetric monoidal groupoid with objects the finite sets, with automorphism groups the symmetric groups and with no morphisms between distinct objects, the monoidal structure is given by the disjoint union $\sqcup$. Quillen's bracket construction $\mathfrak{U}\Sigma$ is equivalent to the category FI of finite sets and injections used in [7]. For R a commutative ring, R-$\mathfrak{Mod}$ denotes the category of R-modules. We prove the following results.

***Theorem B* (Proposition 3.9, Proposition 3.10, Example 3.15).** *Let $\mathbb{K}$ be a field of characteristic zero and d be a natural number. Considering functors $F : FI \to \mathbb{K}$-$\mathfrak{Mod}$, we have:*

1. *For n a natural number, we respectively denote by $\mathbf{B}_n$ the braid group on n strands and by $\mathbf{PB}_n$ the pure braid group on n strands. Then, $H_d\left(\mathbf{B}_\infty, F_\infty\right) \cong \underset{n \in FI}{Colim}\left(H_d\left(\mathbf{PB}_n, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} F(n)\right)$.*

2. *$H_d\left(\Gamma_{\infty,1}^{\infty}, F_\infty\right) \cong \underset{n \in FI}{Colim}\left[\bigoplus_{k+l=d}\left(H_k\left(\Gamma_{n,1}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n}, \mathbb{K}\right)\right) \underset{\mathbb{K}}{\otimes} F(n)\right]$, where $\Gamma_{g,1}^{s}$ denotes the isotopy classes of diffeomorphisms permuting the marked points and restricting to the identity on the boundary component of a compact connected orientable surface with one boundary component, genus $g \geq 0$ and $s \geq 0$ marked points. It follows from Madsen-Weiss theorem [27] that $H_{2k+1}\left(\Gamma_{\infty,1}^{\infty}, F_\infty\right) = 0$ for all natural numbers k.*

3. *$H_d\left(Aut\left(\left(\mathbb{Z}^{*k}\right)^{\times\infty}\right), F_\infty\right) = 0$ for a fixed natural number $k \geq 2d+1$ (where $*$ denotes the free product of groups).*

Moreover, we may make further explicit calculations for specific polynomial FI-modules F using the formulas of Theorem B, although it generally requires some non-trivial extra work: the key point is to understand the FI-module structure of the left-hand terms in the formula. Namely, the action of the symmetric group $\mathfrak{S}_n$ on the homology group $H_q\left(\mathbf{PB}_n, \mathbb{K}\right)$ is studied in [7, Example 5.1.A] using the computations of [1] and the one on $\bigoplus_{k+l=d}\left(H_k\left(\Gamma_{-,1}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times -}, \mathbb{K}\right)\right)$ is induced by the action of the symmetric groups on the homology of $\mathbb{C}P^\infty$. Therefore we can understand the FI-module structure of the associated pointwise tensor product and then compute the colimit with respect to FI.

The proof of Theorem B requires a splitting result for the twisted stable homology for some families of groups: this decomposition consists in the graded direct sum of tensor products of the homology of an associated category with the stable homology with constant coefficients. Namely, we assume that the category $(\mathfrak{U}\mathcal{G}, \natural, 0)$ is pre-braided homogeneous (we refer the reader to Section 1 for an introduction to these notions) such that the unit 0 is an initial object. For a functor $F : \mathfrak{U}\mathcal{G} \to \mathbf{Ab}$, we denote by $H_*\left(\mathfrak{U}\mathcal{G}, F\right)$ the homology of the category $\mathfrak{U}\mathcal{G}$ with coefficient in F (we refer the reader to the papers [12, Section 2] and [9, Appendice A] for an introduction to this last notion). We prove the following statement.

***Theorem C* (Theorem 3.2) .** *Let $\mathbb{K}$ be a field. For all functors $F : \mathfrak{U}\mathcal{G} \to \mathbb{K}$-$\mathfrak{Mod}$, we have a natural isomorphism of $\mathbb{K}$-modules:*

$$H_*\left(G_\infty, F_\infty\right) \cong \bigoplus_{k+l=*}\left(H_k\left(G_\infty, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\mathfrak{U}\mathcal{G}, F\right)\right).$$

If the groupoid $\mathcal{G}$ is symmetric monoidal, then Theorem C recovers the previous analogous results [9, Propositions 2.22, 2.26].

The paper is organized as follows. In Section 1, we recall notions on Quillen's bracket construction, pre-braided monoidal categories and homogeneous categories. In Section 2, after setting up the general framework for the families of groups we will deal with and applying Lyndon-Hochschild-Serre spectral sequence, we prove the various results of Theorem A. In Section 3, the first part is devoted to the proof of the decomposition result Theorem C. Then we deal with the twisted stable homology for mapping class groups with non-trivial finite quotient groups and prove Theorem B.

General notations. We fix R a commutative ring and $\mathbb{K}$ a field throughout this work. We denote by R-$\mathfrak{Mod}$ the categories of R-modules.

Let $\mathfrak{Cat}$ denote the category of small categories. Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$. We use the abbreviation $Obj(\mathfrak{C})$ to denote the objects of $\mathfrak{C}$. For $\mathfrak{D}$ a category, we denote by $\mathbf{Fct}(\mathfrak{C},\mathfrak{D})$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$. If 0 is initial object in the category $\mathfrak{C}$, then we denote by $\iota_A : 0 \to A$ the unique morphism from 0 to A. The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and of which the morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{G}\mathfrak{r} : \mathfrak{Cat} \to \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid.

We take the convention that the set of natural numbers $\mathbb{N}$ is the set of nonnegative integers $\{0, 1, 2, \ldots\}$. We denote by $(\mathbb{N}, \leq)$ the category of natural numbers considered as a directed set. For all natural numbers n, we denote by γ_n the unique element of $Hom_{(\mathbb{N},\leq)}(n, n+1)$. For all natural numbers n' such that $n' \geq n$, we denote by $\gamma_{n,n'} : n \to n'$ the unique element of $Hom_{(\mathbb{N},\leq)}(n, n')$, composition of the morphisms $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \cdots \circ \gamma_{n+1} \circ \gamma_n$. The addition defines a strict monoidal structure on $(\mathbb{N}, \leq)$, denoted by $((\mathbb{N}, \leq), +, 0)$.

We denote by $\mathfrak{Gr}$ the category of groups, by $*$ the coproduct in this category, by $\mathbf{Ab}$ the full subcategory of $\mathfrak{Gr}$ of abelian groups and by $\mathfrak{gr}$ the full subcategory of $\mathfrak{Gr}$ of finitely generated free groups. Recall that the free product of groups $*$ defines a monoidal structure over $\mathfrak{gr}$, with the trivial group $0_{\mathfrak{Gr}}$ the unit, denoted by $(\mathfrak{gr}, *, 0_{\mathfrak{Gr}})$. We denote by $\times$ the direct product of groups and by $Aut(G)$ the automorphism group of a group G.

Acknowledgement. The author wishes to thank most sincerely Christine Vespa, Nathalie Wahl and Geoffrey Powell, for their reading, suggestions and advice. He would also like to thank Aurélien Djament, Nariya Kawazumi and Antoine Touzé for the attention they have paid to his work and helpful discussions. Additionally, he would like to thank the anonymous referee for his careful reading, comments and corrections.

Contents

1 Categorical framework **4**

2 Twisted stable homologies of semidirect products **6**
2.1 A general result for the homology of semidirect products . . . 6
2.2 Properties of the twisted coefficients . . . 7
2.3 Applications . . . 9
2.3.1 Mapping class groups of orientable surfaces . . . 9
2.3.2 Automorphisms of free groups with boundaries . . . 13

3 Twisted stable homologies for FI-modules **14**
3.1 General decomposition for the twisted stable homology using functor homology . . . 15
3.2 Framework and first equivalence for stable homology . . . 16
3.3 Applications . . . 18
3.3.1 Braid groups . . . 18
3.3.2 Mapping class group of orientable surfaces . . . 19
3.3.3 Particular right-angled Artin groups . . . 20

1 Categorical framework

This section recollects the notions of Quillen's bracket construction, pre-braided monoidal categories and homogeneous categories for the convenience of the reader. It takes up the framework of [28, Section 1]. For an introduction to braided monoidal categories, we refer to [26, Section XI]. Standardly, a strict monoidal category will be denoted by $(\mathfrak{C}, \natural, 0)$, where $\natural : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ is the monoidal structure and 0 is the monoidal unit. If the category is braided, we denote by $b^{\mathfrak{C}}_{-,-}$ its braiding. **We fix a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ throughout this section.**

Quillen's bracket construction. The following definition is a particular case of a more general construction of [17].

Definition 1.1. [28, Section 1.1] Quillen's bracket construction on the groupoid $\mathfrak{G}$, denoted by $\mathfrak{U}\mathfrak{G}$ is the category defined by:

- Objects: $Obj\,(\mathfrak{U}\mathfrak{G}) = Obj\,(\mathfrak{G})$;
- Morphisms: for A and B objects of $\mathfrak{G}$, $Hom_{\mathfrak{U}\mathfrak{G}}\,(A, B) = \underset{\mathfrak{G}}{colim}\,[Hom_{\mathfrak{G}}\,(-\natural A, B)]$. A morphism from A to B in the category $\mathfrak{U}\mathfrak{G}$ is an equivalence class of pairs (X, f), where X is an object of $\mathfrak{G}$ and $f : X\natural A \to B$ is a morphism of $\mathfrak{G}$; this is denoted by $[X, f] : A \to B$.
- Let $[X, f] : A \to B$ and $[Y, g] : B \to C$ be morphisms in the category $\mathfrak{U}\mathfrak{G}$. Then, the composition in the category $\mathfrak{U}\mathfrak{G}$ is defined by $[Y, g] \circ [X, f] = [Y\natural X, g \circ (id_Y \natural f)]$.

It is clear that the unit 0 of the monoidal structure of the groupoid $(\mathfrak{G}, \natural, 0)$ is an initial object in the category $\mathfrak{U}\mathfrak{G}$ (see [28, Proposition 1.8 (i)]).

Definition 1.2. The strict monoidal category $(\mathfrak{G}, \natural, 0)$ is said to have no zero divisors if for all objects A and B of $\mathfrak{G}$, $A\natural B \cong 0$ if and only if $A \cong B \cong 0$.

Proposition 1.3. *[28, Proposition 1.7] Assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. Then, the groupoid $\mathfrak{G}$ is the maximal subgroupoid of $\mathfrak{U}\mathfrak{G}$.*

Henceforth, we assume that the groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$.

Remark 1.4. Let X be an object of $\mathfrak{G}$. Let $\phi \in Aut_{\mathfrak{G}}\,(X)$. Then, as an element of $Hom_{\mathfrak{U}\mathfrak{G}}\,(X, X)$, we will abuse the notation and write ϕ for $[0, \phi]$.

Finally, we recall the following lemma.

Lemma 1.5. *[28, Proposition 2.4][29, Lemma 1.8] Let $\mathscr{C}$ be a category and F an object of $\mathbf{Fct}\,(\mathfrak{G}, \mathscr{C})$. Assume that for $A, X, Y \in Obj\,(\mathfrak{G})$, there exist assignments $F\,([X, id_{X\natural A}]) : F\,(A) \to F\,(X\natural A)$ such that:*

$$F\,([Y, id_{Y\natural X\natural A}]) \circ F\,([X, id_{X\natural A}]) = F\,([Y\natural X, id_{Y\natural X\natural A}])\,. \tag{1}$$

Then, the assignment $F\,([X, g]) = F\,(g) \circ F\,([X, id_{X\natural A}])$ for $[X, g] \in Hom_{\mathfrak{U}\mathfrak{G}}\,(A, id_{X\natural A})$ defines a functor $F : \mathfrak{U}\mathfrak{G} \to \mathscr{C}$ if and only if for all $A, X \in Obj\,(\mathfrak{G})$, for all $g'' \in Aut_{\mathfrak{G}}\,(A)$ and all $g' \in Aut_{\mathfrak{G}}\,(X)$:

$$F\,([X, id_{X\natural A}]) \circ F\,(g'') = F\,(g'\natural g'') \circ F\,([X, id_{X\natural A}])\,. \tag{2}$$

Pre-braided monoidal categories. Assuming that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is braided, Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$ also inherits a strict monoidal structure (see Proposition 1.8). Beforehand, we recall the notion of pre-braided category, introduced by Randal-Williams and Wahl in [28, Section 1].

Definition 1.6. [28, Definition 1.5] Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal category such that the unit 0 is initial. We say that the monoidal category $(\mathfrak{C}, \natural, 0)$ is pre-braided if:

- The maximal subgroupoid $\mathscr{G}\mathfrak{r}\,(\mathfrak{C}, \natural, 0)$ is a braided monoidal category, where the monoidal structure is induced by that of $(\mathfrak{C}, \natural, 0)$.

- For all objects A and B of $\mathfrak{C}$, the braiding associated with the maximal subgroupoid $b^{\mathfrak{C}}_{A,B} : A\natural B \longrightarrow B\natural A$ satisfies:
$$b^{\mathfrak{C}}_{A,B} \circ (id_A \natural \iota_B) = \iota_B \natural id_A : A \longrightarrow B\natural A. \tag{3}$$

Remark 1.7. A braided monoidal category is always pre-braided but the converse is false. Indeed, for a pre-braided monoidal category, the opposite of condition (3) (i.e. $b^{\mathfrak{C}}_{A,B} \circ (\iota_B \natural id_A) = id_A \natural \iota_B$) does not hold generally speaking, whereas this is a necessary property for a braided monoidal category. For instance, the category $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ is pre-braided monoidal but not braided since $b^{\boldsymbol{\beta}}_{1,2} \circ (\iota_1 \natural id_2) \neq id_2 \natural \iota_1$ (see [28, Remark 5.24] if more details are required).

Finally, we give the effect of Quillen's bracket construction over the strict braided monoidal groupoid $(\mathfrak{G}, \natural, 0)$.

Proposition 1.8. *[28, Proposition 1.8] Suppose that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. If the groupoid $(\mathfrak{G}, \natural, 0)$ is braided, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is pre-braided monoidal. If the groupoid $(\mathfrak{G}, \natural, 0)$ is symmetric, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is symmetric monoidal.*

The monoidal structure on the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is defined on objects as for $(\mathfrak{G}, \natural, 0)$ and defined on morphisms by letting, for $[X, f] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, B)$ and $[Y, g] \in Hom_{\mathfrak{U}\mathfrak{G}}(C, D)$:

$$[X, f] \natural [Y, g] = \left[X\natural Y, (f\natural g) \circ \left(id_X \natural \left(b^{\mathfrak{G}}_{A,Y}\right)^{-1} \natural id_C\right)\right].$$

In particular, the canonical functor $\mathfrak{G} \to \mathfrak{U}\mathfrak{G}$ (see Remark 1.4) is monoidal.

Homogeneous categories. The notion of homogeneous category is introduced by Randal-Williams and Wahl in [28, Section 1], inspired by the set-up of Djament and Vespa in [9, Section 1.2]. With two additional assumptions, Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$ from a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is endowed with an homogeneous category structure. This type of category is very useful to deal with homological stability with twisted coefficients questions (see [28]) or to work on the stable homology with twisted coefficient (see [9], [10] and Section 3.1).

Let $(\mathfrak{C}, \natural, 0)$ be a small strict monoidal category in which the unit 0 is also initial. For all objects A and B of $\mathfrak{C}$, we consider the morphism $\iota_A \natural id_B : 0\natural B \longrightarrow A\natural B$ and a set of morphisms:

$$Fix_A(B) = \{\phi \in Aut(A\natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\}.$$

Since $(\mathfrak{C}, \natural, 0)$ is assumed to be small, $Hom_{\mathfrak{C}}(A, B)$ is a set and $Aut_{\mathfrak{C}}(B)$ defines a group (with composition of morphisms as the group product). The group $Aut_{\mathfrak{C}}(B)$ acts by post-composition on $Hom_{\mathfrak{C}}(A, B)$:

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(B) \times Hom_{\mathfrak{C}}(A, B) & \longrightarrow & Hom_{\mathfrak{C}}(A, B). \\ (\phi, f) & \longmapsto & \phi \circ f \end{array}$$

Definition 1.9. Let $(\mathfrak{C}, \natural, 0)$ be a small strict monoidal category where the unit 0 is initial. We consider the following axioms:

- (**H1**): for all objects A and B of the category $\mathfrak{C}$, the action by post-composition of $Aut_{\mathfrak{C}}(B)$ on $Hom_{\mathfrak{C}}(A, B)$ is transitive.
- (**H2**): for all objects A and B of the category $\mathfrak{C}$, the map $Aut_{\mathfrak{C}}(A) \to Aut_{\mathfrak{C}}(A\natural B)$ sending $f \in Aut_{\mathfrak{C}}(A)$ to $f\natural id_B$ is injective with image $Fix_A(B)$.

The category $(\mathfrak{C}, \natural, 0)$ is homogeneous if it satisfies the axioms (**H1**) and (**H2**).

As a consequence of the axioms (**H1**) and (**H2**), we deduce that:

Lemma 1.10. *If $(\mathfrak{C}, \natural, 0)$ is a homogeneous category, then $Hom_{\mathfrak{C}}(B, A\natural B) \cong Aut_{\mathfrak{C}}(A\natural B) / Aut_{\mathfrak{C}}(A)$ for all objects A and B and where $Aut_{\mathfrak{C}}(A)$ acts on $Aut_{\mathfrak{C}}(A\natural B)$ by precomposition.*

We now give the two additional properties so that if a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ satisfy them, then Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$ is homogeneous.

Definition 1.11. Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal category. We define two assumptions.

- $(\mathbf{C})$: for all objects A, B and C of $\mathfrak{C}$, if $A\natural C \cong B\natural C$ then $A \cong B$.
- $(\mathbf{I})$: for all objects A, B of $\mathfrak{C}$, the morphism $Aut_{\mathfrak{C}}(A) \to Aut_{\mathfrak{C}}(A\natural B)$ sending $f \in Aut_{\mathfrak{C}}(A)$ to $f\natural id_B$ is injective.

Theorem 1.12. *[28, Theorem 1.10] Let $(\mathfrak{G}, \natural, 0)$ be a braided monoidal groupoid with no zero divisors. If the groupoid $\mathfrak{G}$ satisfies $(\mathbf{C})$ and $(\mathbf{I})$, then $\mathfrak{U}\mathfrak{G}$ is homogeneous.*

2 Twisted stable homologies of semidirect products

This section introduces a general method to compute the stable homology with twisted coefficients using semidirect product structures arising naturally from the families of mapping class groups. We first establish the general result of Corollary 2.3 for the homology of semidirect products with twisted coefficients. These results are finally applied in Section 2.3 to compute explicitly some homology groups with twisted coefficients for mapping class groups of orientable surfaces and automorphisms of free groups with boundaries. Beforehand, we take this opportunity to introduce the following terminology:

Definition 2.1. A family of groups is a functor $\mathbf{G}_{-} : (\mathbb{N}, \leq) \longrightarrow \mathfrak{Gr}$ such that for all natural numbers n, $\mathbf{G}_{-}(\gamma_n) : \mathbf{G}_n \hookrightarrow \mathbf{G}_{n+1}$ is an injective group morphism.

2.1 A general result for the homology of semidirect products

First, we present some properties for the homology with twisted coefficients for a semidirect product and prove the general statement of Corollary 2.3.

Let $\mathcal{Q}$ be a groupoid with objects indexed by the natural numbers. An object of $\mathcal{Q}$ is thus denoted by $\underline{n}$, where n is its corresponding indexing natural number. We denote by $Aut_{\mathcal{Q}}(\underline{n}) = Q_n$ the automorphism groups and assume that there are no morphisms between distinct objects. We assume that there exists a family of groups $K_{-} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and a functor $\mathcal{A}_{\mathcal{Q}} : \mathcal{Q} \to \mathfrak{Gr}$ such that K_n is any free goup and $\mathcal{A}_{\mathcal{Q}}(\underline{n}) = K_n$ for all natural numbers n. We denote by $\mathcal{A}_{\mathcal{Q},n} : Q_n \to Aut_{\mathfrak{Gr}}(K_n)$ the group morphism induced by the functor $\mathcal{A}_{\mathcal{Q}}$ for each natural number n. Hence, we form the split short exact sequence:

$$1 \longrightarrow K_n \xrightarrow{k_n} K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n \xrightarrow{q_n} Q_n \longrightarrow 1 \tag{4}$$

and we denote by $s_n : Q_n \to K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n$ the canonical splitting of q_n. For all natural numbers n, we fix M_n a $R\left[K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n\right]$-module. We abuse the notation and write M_n for the restriction of M_n from $K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n$ to K_n. Then we can construct the following long exact sequence in homology, analogous to the Gysin sequence for homology:

Proposition 2.2. *For M_n an $R\left[K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n\right]$-module, the short exact sequence (4) induces a long exact sequence:*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{*+1}(Q_n, H_0(K_n, M_n)) \arrow[d, "d^2_{*+1,0}"] & & & \\
& H_{*-1}(Q_n, H_1(K_n, M_n)) \arrow[r, "\varphi_*"] & H_*\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) \arrow[r, "\psi_*"] & H_*(Q_n, H_0(K_n, M_n)) \arrow[d, "d^2_{*,0}"] & \\
& & & H_{*-2}(Q_n, H_1(K_n, M_n)) \arrow[r] & \cdots
\end{tikzcd} \tag{5}$$

where $\left\{d^2_{p,q}\right\}_{p,q\in\mathbb{N}}$ denote the differentials of the second page of the Lyndon-Hochschild-Serre spectral sequence associated with the short exact sequence (4).

Proof. Applying the Lyndon-Hochschild-Serre spectral sequence to the short exact sequence (4), we obtain the following convergent first quadrant spectral sequence:

$$E^2_{pq} : H_p\left(Q_n, H_q\left(K_n, M_n\right)\right) \Longrightarrow H_{p+q}\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right). \tag{6}$$

Since K_n is a free group, $H_q\left(K_n, M_n\right) = 0$ for $q \geq 2$. The result is a classical consequence of the fact that the spectral sequence (6) has only two rows. In particular, the map φ_* is defined by the composition:

$$H_{*-1}\left(Q_n, H_1\left(K_n, M_n\right)\right) \twoheadrightarrow H_{*-1}\left(Q_n, H_1\left(K_n, M_n\right)\right)/\mathrm{Im}\left(d^2_{*+1,0}\right) \hookrightarrow H_*\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right);$$

the map ψ_* is the coinflation map $\mathrm{Coinf}^{Q_n}_{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}\left(M_n\right)$, induced by the composition:

$$H_*\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) \twoheadrightarrow Ker\left(d^2_{*,0}\right) \hookrightarrow H_*\left(Q_n, H_0\left(K_n, M_n\right)\right).$$

□

Corollary 2.3. *Let n be a natural number. Assume that the free group K_n acts trivially on the R-module M_n. Then, for all natural numbers $q \geq 1$:*

$$H_{q-1}\left(Q_n, H_1\left(K_n, R\right) \underset{R}{\otimes} M_n\right) \oplus H_q\left(Q_n, M_n\right) \cong H_q\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right). \tag{7}$$

Proof. As M_n is a trivial K_n-module:

$$H_1\left(K_n, M_n\right) \cong H_1\left(K_n, R\right) \underset{R}{\otimes} M_n \text{ and } H_0\left(K_n, M_n\right) \cong M_n,$$

and the coinflation map $\psi_* = \mathrm{Coinf}^{Q_n}_{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}\left(M_n\right)$ is equal to the corestriction map $\mathrm{Cores}^{Q_n}_{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}\left(M_n\right)$. Hence, denoting by $H_*\left(q_n, M_n\right)$ the map induced in homology by $q_n : K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n \to Q_n$, we deduce that $\psi_* = H_*\left(q_n, M_n\right)$. By the functoriality of group homology, the splitting $s_n : Q_n \to K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n$ of q_n induces a splitting in homology $H_*\left(s_n, M_n\right)$ of $H_*\left(q_n, M_n\right)$. Hence, $H_*\left(p_n, M_n\right)$ is an epimorphism and a fortiori $Ker\left(d^2_{*,0}\right) \cong H_*\left(Q_n, M_n\right)$. Therefore, $d^2_{*,0} = 0$ and the exact sequence (5) gives a split short exact sequence of abelian groups for every $q \geq 1$:

$$1 \longrightarrow H_{q-1}\left(Q_n, H_1\left(K_n, R\right) \underset{R}{\otimes} M_n\right) \xrightarrow{\varphi_q} H_q\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) \xrightarrow{H_q(q_n, M_n)} H_q\left(Q_n, M_n\right) \longrightarrow 1. \tag{8}$$

□

2.2 Properties of the twisted coefficients

Our aim here is to study the twisted coefficients $H_1\left(K_n, R\right) \underset{R}{\otimes} M_n$ appearing in Corollary 2.3 so as to prove Lemma 2.9. This last result will be useful to prove Theorem 2.24. We now make the following further assumptions that:

- the groupoid $\mathcal{Q}$ is a braided strict monoidal category (we denote by $(\mathcal{Q}, \natural, 0)$ the monoidal structure);
- there exists a free group K such that $K_n \cong K^{*n}$ and that that $K_-\left(\gamma_n\right) = \iota_K * id_{K_n}$ (where γ_n denotes the unique element of $Aut_{(\mathbb{N},\leq)}\left(n, n+1\right)$) for all natural numbers n;
- the functor $\mathcal{A}_{\mathcal{Q}}$ takes values in the subcategory $\mathfrak{gr} \subset \mathfrak{Gr}$ and defines a strict monoidal functor $(\mathcal{Q}, \natural, 0) \to (\mathfrak{gr}, *, 0)$.

These assumptions allow to define the functor K_- on the category $\mathfrak{U}\mathcal{Q}$:

Lemma 2.4. *Assigning $\mathcal{A}_{\mathcal{Q}}\left(\left[\underline{1}, id_{\underline{n+1}}\right]\right) = K_{-}\left(\gamma_n\right)$ for all natural numbers n, we define a functor $\mathcal{A}_{\mathcal{Q}} : \mathfrak{U}\mathcal{Q} \to \mathfrak{gr}$.*

Proof. We use Lemma 1.5 to prove this result: namely, we show that relations (1) and (2) of this lemma are satisfied. It follows from the fact that K_{-} is a functor on $(\mathbb{N}, \leq)$, that the relation (1) of Lemma 1.5 is satisfied by $\mathcal{A}_{\mathcal{Q}}$. Let n and n' be natural numbers such that $n' \geq n$, let $q \in Q_n$ and $q' \in Q_{n'}$. We denote by $e_{K_{n'}}$ the neutral element of $K_{n'}$. Since $\mathcal{A}_{\mathcal{Q}}$ is monoidal, we compute for all $k \in K_n$:

$$\begin{aligned}
\left(\mathcal{A}_{\mathcal{Q}}\left(q' \natural q\right) \circ \mathcal{A}_{\mathcal{Q}}\left(\left[\underline{n'}, id_{\underline{n'+n}}\right]\right)\right)(k) &= \left(\mathcal{A}_{\mathcal{Q}}\left(q'\right) * \mathcal{A}_{\mathcal{Q}}\left(q\right)\right)\left(e_{K_{n'}} * k\right) \\
&= e_{K_{n'}} * \mathcal{A}_{\mathcal{Q}}\left(q\right)(k) \\
&= \left(\mathcal{A}_{\mathcal{Q}}\left(\left[\underline{n'}, id_{\underline{n'+n}}\right]\right) \circ \mathcal{A}_{\mathcal{Q}}\left(q\right)\right)(k).
\end{aligned}$$

Hence, the relation (2) of Lemma 1.5 is satisfied by $\mathcal{A}_{\mathcal{Q}}$. □

Recollections on strong polynomial functors. We deal here with the concept of strong and very strong polynomial functors, which will be useful to prove Theorems 2.17 and 2.24. We refer the reader to [30, Section 3] for a complete introduction to these notions for pre-braided monoidal categories as source category, extending the previous framework due to Djament and Vespa in [11] for symmetric monoidal categories. They also are particular case of coefficient systems of finite degree introduced by Randal-Williams and Wahl in [28], thus providing a natural setting to study homological stability.

From now we fix $(\mathfrak{M}, \natural, 0)$ a pre-braided strict monoidal category such that the monoidal unit 0 is an initial object. For all objects X of $\mathfrak{M}$, the monoidal structure $\natural$ defines the endofunctor $X \natural - : \mathfrak{M} \to \mathfrak{M}$, which sends the object Y to the object $X \natural Y$. We define the translation functor $\tau_X : \mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ to be the endofunctor obtained by precomposition by $X \natural -$.

For all objects F of $\mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ and all objects X of $\mathfrak{M}$, we denote by $i_X(F) : F \to \tau_X(F)$ the natural transformation induced by the unique morphism $\left[X, id_X\right] : 0 \to X$ of $\mathfrak{M}$. This induces $i_X : Id_{\mathbf{Fct}(\mathfrak{M}, R\text{-}\mathfrak{Mod})} \to \tau_X$ a natural transformation of $\mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$. Since the category $\mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ is abelian (since the target category $R\text{-}\mathfrak{Mod}$ is abelian), the kernel and cokernel of the natural transformation i_X exist. We define the functors $\kappa_X = \ker\left(i_X\right)$ and $\delta_X = \operatorname{coker}\left(i_X\right)$. Then:

Definition 2.5. We recursively define on $d \in \mathbb{N}$ the categories $\mathcal{P}ol_d^{strong}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ and $\mathcal{VP}ol_d\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ of strong and very strong polynomial functors of degree less than or equal to d to be the full subcategories of $\mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ as follows:

1. If $d < 0$, $\mathcal{P}ol_d^{strong}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right) = \mathcal{VP}ol_d\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right) = \{0\}$;
2. if $d \geq 0$, the objects of $\mathcal{P}ol_d^{strong}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ are the functors F such that the functor $\delta_X(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ for all objects X of $\mathfrak{M}$; the objects of $\mathcal{VP}ol_d\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ are the objects F of $\mathcal{P}ol_d\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ such that $\kappa_X(F) = 0$ and the functor $\delta_X(F)$ is an object of $\mathcal{VP}ol_{d-1}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ for all objects X of $\mathfrak{M}$.

For an object F of $\mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ which is strong of degree less than or equal to $n \in \mathbb{N}$, the smallest natural number $d \leq n$ for which F is an object of $\mathcal{P}ol_d^{strong}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ is called the strong degree of F. If in addition F is strong very strong polynomial, its strong degree is also the smallest natural number $d \leq n$ for which F is an object of $\mathcal{VP}ol_d\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ and is then also called the very strong degree of F.

Remark 2.6. If in addition the unit 0 of the monoidal structure $(\mathfrak{M}, \natural, 0)$ is a terminal object, the evanescence functors automatically vanish and a fortiori the notions of strong and very strong polynomial functors are equivalent.

Furthermore, we have the following property which will be useful for our further work. The result for strong polynomial functors is already established in [30, Proposition 3.8] and previously for symmetric monoidal categories [11, Proposition 1.7].

Proposition 2.7. *Let $\mathfrak{M}'$ be another pre-braided strict monoidal category such that such that its monoidal unit is an initial object. Let $\alpha : \mathfrak{M} \to \mathfrak{M}'$ be a strong monoidal functor. Then, the precomposition by α provide functors $\mathcal{VP}ol_n\left(\mathfrak{M}', R\text{-}\mathfrak{Mod}\right) \to \mathcal{VP}ol_n\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$ which preserves the degree of polynomiality.*

Proof. Let X an object of $\mathfrak{M}$ and F be an object of $\mathbf{Fct}\left(\mathfrak{M}, R\text{-}\mathfrak{Mod}\right)$. Since α is strong monoidal, we deduce that there is a natural equivalence $\tau_X\left(F\circ\alpha\right)\cong\left(\tau_{\alpha(X)}F\right)\circ\alpha$. It is a standard fact that the precomposition by α is an exact functor. Then the universal properties of the kernel and cokernel imply that there are natural equivalences $\delta_X\left(F\circ\alpha\right)\cong\left(\delta_{\alpha(X)}F\right)\circ\alpha$ and $\kappa_X\left(F\circ\alpha\right)\cong\left(\kappa_{\alpha(X)}F\right)\circ\alpha$. The results then follow from a straightforward recursion on the degree of polynomiality. □

First homology functor. Recall that the homology group $H_1\left(-,R\right)$ defines a functor from the category $\mathfrak{Gr}$ to the category $R\text{-}\mathfrak{Mod}$ (see for example [5, Section 8]). Hence, we introduce the following functor:

Definition 2.8. The homology groups $\left\{H_1\left(K_n,R\right)\right\}_{n\in\mathbb{N}}$ assemble to define a functor $H_1\left(\mathcal{A}_Q,R\right):\mathfrak{U}\mathcal{Q}\to R\text{-}\mathfrak{Mod}$ by the composite $H_1\left(-,R\right)\circ\mathcal{A}_Q$. It is called the first homology functor of $\mathcal{A}_Q$.

If $R=\mathbb{Z}$ and K_n is finitely generated for all natural numbers n, the target category of $H_1\left(\mathcal{A}_Q,R\right)$ is the full subcategory $\mathbf{ab}\subset\mathbf{Ab}$ of finitely generated abelian groups. Let m be a natural number. We then define a functor $H_1\left(\mathcal{A}_Q,\mathbb{Z}\right)^{\otimes m}:\mathfrak{U}\mathcal{Q}\to\mathbf{ab}$ by the composite $-^{\otimes m}\circ H_1\left(\mathcal{A}_Q,\mathbb{Z}\right)$ where $-^{\otimes m}:\mathbf{ab}\to\mathbf{ab}$ sends an object G to $G^{\otimes m}$.

Lemma 2.9. *If the groups K_n are finitely generated free for all n, then the functor $H_1\left(\mathcal{A}_Q,\mathbb{Z}\right)^{\otimes m}$ is very strong polynomial of degree m.*

Proof. Recall that the free product gives a symmetric monoidal structure $\left(\mathfrak{Gr},*,0_{\mathfrak{Gr}}\right)$ (which restricts to $\mathfrak{gr}$), that the direct sum defines a symmetric monoidal structure $\left(\mathbf{Ab},\oplus,0_{\mathfrak{Gr}}\right)$ (which restricts to $\mathbf{ab}$) and that symmetric monoidal categories are particular cases of pre-braided monoidal ones. By the result of the homology of a free product of groups (see for example [32, Corollary 6.2.10]), the first homology group is a strong monoidal functor $H_1\left(-,R\right):\left(\mathfrak{Gr},*,0_{\mathfrak{Gr}}\right)\to\left(\mathbf{Ab},\oplus,0_{\mathfrak{Gr}}\right)$ and a fortiori so is the restriction $H_1\left(-,R\right):\mathfrak{gr}\to\mathbf{ab}$. As $\mathcal{A}_Q$ is a strict monoidal functor, then $H_1\left(\mathcal{A}_Q,\mathbb{Z}\right)$ is a strong monoidal functor. This is a well-known fact that the m-th tensor power functor $-^{\otimes m}:\mathbf{ab}\to\mathbf{ab}$ is very strong polynomial of degree m (see [9, Appendice A] for example). Then the result follows from Proposition 2.7. □

Pointwise tensor product. We finally recall the following result, used to prove Theorem 2.24. For $\mathfrak{M}$ a pre-braided strict monoidal category such that such that its monoidal unit is an initial object, the pointwise tensor product of two objects of $\mathbf{Fct}\left(\mathfrak{M},R\text{-}\mathfrak{Mod}\right)$ defines an object of $\mathbf{Fct}\left(\mathfrak{M},R\text{-}\mathfrak{Mod}\right)$, assigning

$$\left(M\underset{R}{\otimes}M'\right)\left(X\right)=M\left(X\right)\underset{R}{\otimes}M'\left(X\right)$$

for $M,M'\in\mathbf{Fct}\left(\mathfrak{M},R\text{-}\mathfrak{Mod}\right)$ and for all objects X of $\mathfrak{M}$.

Lemma 2.10. *If M and M' are strong polynomial functors, then $M\underset{R}{\otimes}M'$ is a strong polynomial functor.*

Proof. We fix an object X of $\mathfrak{M}$. Since the translation functor τ_X commutes with all limits, and as a colimit of a natural transformation between Id and τ_X, the functor δ_X commutes with the (pointwise) product $\times$. Let d be the largest of the two strong polynomial degrees. Hence, $\underbrace{\delta_X\cdots\delta_X}_{d+1\text{ times}}\left(M\times M'\right)=0$ and therefore $M\underset{R}{\otimes}M'$ is a strong polynomial functor of degree less than or equal to $d+1$. □

2.3 Applications

Many families of mapping class groups fall within the framework of Section 2.1. Corollary 2.3 is the key result to compute the homology with twisted coefficients for these families of groups.

2.3.1 Mapping class groups of orientable surfaces

Let $\Sigma_{g,i}^{s}$ denote a smooth compact connected orientable surface with (orientable) genus $g\in\mathbb{N}$, $s\in\mathbb{N}$ marked points and $i\in\{1,2\}$ boundary components, with $I:[-1,1]\to\partial\Sigma_{g,i}^{s}$ a parametrized interval in the boundary if $i=1$ and $p=0\in I$ a basepoint. We denote by $\mathbf{\Gamma}_{g,1}^{s}$ (respectively $\mathbf{\Gamma}_{g,1}^{[s]}$) the isotopy classes of diffeomorphisms of $\Sigma_{g,1}^{s}$ preserving the orientation, restricting to the identity on a neighbourhood of the parametrized interval I and permuting (respectively fixing) the marked points. If $s=0$, we omit it from the notation. Recall that, up to isotopy, fixing the interval I is the same as fixing the whole boundary component pointwise. When there is no ambiguity, we omit the parametrized interval I from the notation.

Let $\natural$ be the boundary connected sum along half of each interval I. For two decorated surfaces $\Sigma_{g_1,1}^{s_1}$ and $\Sigma_{g_2,1}^{s_2}$, the boundary connected sum $\Sigma_{g_1,1}^{s_1}\natural\Sigma_{g_2,1}^{s_2}$ is defined as the surface obtained from gluing $\Sigma_{g_1,1}^{s_1}$ and $\Sigma_{g_2,1}^{s_2}$ along the half-interval I_1^+ and the half-interval I_2^-, and $I_1\natural I_2 = I_1^- \bigcup I_2^+$. The homeomorphisms being the identity on a neighbourhood of the parametrized intervals I_1 and I_2, we canonically extend the diffeomorphisms of $\Sigma_{g_1,1}^{s_1}$ and $\Sigma_{g_2,1}^{s_2}$ to $\Sigma_{g_1,1}^{s_1}\natural\Sigma_{g_2,1}^{s_2}$. For completeness, we refer to [28, Section 5.6.1], for technical details.

We denote by $\mathbf{\Gamma}_{g,2}$ the isotopy classes of diffeomorphisms of $\Sigma_{g,2}^0$ preserving the orientation and fixing the boundary components pointwise. Recall that R is a commutative ring and we assume that the various mapping class groups act trivially on it.

The following result is an essential tool for our work:

Theorem 2.11. *[2] Let $g\geq 1$, $s\geq 0$ be natural numbers and x be a marked point in the interior of $\Sigma_{g,1}^s$. Forgetting x induces a map $\omega_s : \mathbf{\Gamma}_{g,1}^{[s+1]} \to \mathbf{\Gamma}_{g,1}^{[s]}$ which defines the following short exact sequence:*

$$1 \longrightarrow \pi_1\left(\Sigma_{g,1}^s, x\right) \longrightarrow \mathbf{\Gamma}_{g,1}^{[s+1]} \xrightarrow{\omega_s} \mathbf{\Gamma}_{g,1}^{[s]} \longrightarrow 1. \tag{9}$$

Gluing a disc with a marked point $\Sigma_{0,1}^1$ on the boundary component without the interval I induces the following short exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbf{\Gamma}_{g,2} \xrightarrow{\rho} \mathbf{\Gamma}_{g,1}^1 \longrightarrow 1. \tag{10}$$

For all natural numbers g and s, we denote by $a_{\Sigma_{g,1}^s}^x$ the action of the mapping class group $\mathbf{\Gamma}_{g,1}^{[s]}$ on the fundamental group $\pi_1\left(\Sigma_{g,1}^s, x\right)$.

Lemma 2.12. *The short exact sequence (9) splits.*

Proof. We denote by $\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^n\right)$ the space of diffeomorphisms of the surface $\Sigma_{g,1}^n$ which fix the boundary pointwise and fix the marked points. We recall that the exact sequence (9) is constructed from the long exact sequence of homotopy groups with the locally trivial fibration $\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^{1+s}\right) \overset{\widehat{\omega}_s}{\hookrightarrow} \mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^s\right) \to \Sigma_{g,1} \setminus \{s \text{ points}\}$ using the fact that $\pi_1\left(\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^s\right)\right) = 0$ by [16, Théorème 1]. Namely the fibre $\widehat{\omega}_s$ is defined by forgetting that the additional marked point is fixed and induces the morphism $\omega_s : \mathbf{\Gamma}_{g,1}^{[s+1]} \to \mathbf{\Gamma}_{g,1}^{[s]}$.

We consider $\mathrm{Emb}\left(\Sigma_{0,1}^1, \Sigma_{0,1}^1\natural\Sigma_{g,1}^s\right)$ the space of embeddings taking $I_{\Sigma_{0,1}^1}^-$ to $I_{\Sigma_{0,1}^1\natural\Sigma_{g,1}^s}^-$ and such that the complement of $\Sigma_{0,1}^1$ in $\Sigma_{0,1}^1\natural\Sigma_{g,1}^s \simeq \Sigma_{g,1}^{1+s}$ is diffeomorphic to $\Sigma_{g,1}^s$. Using the parameterised isotopy extension theorem [6, II 2.2.2 Corollaire 2], there is a fibration sequence

$$\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^s\right) \overset{\varrho_s}{\hookrightarrow} \mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{0,1}^1\natural\Sigma_{g,1}^s\right) \to \mathrm{Emb}\left(\left(\Sigma_{0,1}^1\right), \left(\Sigma_{0,1}^1\natural\Sigma_{g,1}^s\right)\right),$$

which long exact sequence of homotopy groups defines a morphism $\pi_0\left(\varrho_s\right) : \mathbf{\Gamma}_{g,1}^{[s]} \to \mathbf{\Gamma}_{g,1}^{[s+1]}$. More precisely, for all $\varphi \in \mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^s\right)$, the morphism ϱ_s is explicitly defined by $\varrho_s\left(\varphi\right) = id_{\Sigma_{0,1}^1}\natural\varphi$. Implicitly, we identify $\Sigma_{g,1}^s$ with $\Sigma_{0,1}^0\natural\Sigma_{g,1}^s$: there is a self-embedding $e : \Sigma_{g,1}^s \hookrightarrow \Sigma_{g,1}^s$ (the complement of whose image is a disc). Then $\widehat{\omega}_s \circ \varrho_s\left(\varphi\right)$ is defined on the image of e by $e\circ\varphi\circ e^{-1}$ and by the identity on the complement of the image of e. Therefore the choice of an isotopy of self-embeddings from e to the identity of $\Sigma_{g,1}^s$ induces a homotopy from $\widehat{\omega}_s \circ \varrho_s$ to the identity of $\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma_{g,1}^s\right)$. Hence, $\widehat{\omega}_s \circ \varrho_s\left(\varphi\right)$ is isotopic to φ: the composition $\widehat{\omega}_s \circ \varrho_s$ is thus isotopic to the identity. We deduce that $\pi_0\left(\varrho_s\right) : \mathbf{\Gamma}_{g,1}^{[s]} \to \mathbf{\Gamma}_{g,1}^{[s+1]}$ is a 1-sided inverse of the map $\omega_s : \mathbf{\Gamma}_{g,1}^{[s+1]} \to \mathbf{\Gamma}_{g,1}^{[s]}$. Therefore the morphism $\pi_0\left(\varrho_s\right)$ is injective and provides a splitting of the exact sequence (9), which defines an isomorphism $\mathbf{\Gamma}_{g,1}^{[s+1]} \cong \pi_1\left(\Sigma_{g,1}^s, x\right) \underset{a_{\Sigma_{g,1}^s}^x}{\rtimes} \mathbf{\Gamma}_{g,1}^{[s]}$. $\square$

Hence, applying Corollary 2.3 to this situation, we obtain:

Proposition 2.13. *Let n, s and $q \geq 1$ be natural numbers. Let M_n be a $R\left[\mathbf{\Gamma}_{n,1}^{[s+1]}\right]$-module on which $\pi_1\left(\Sigma_{n,1}^{s}, x\right)$ acts trivially. Then:*

$$H_q\left(\mathbf{\Gamma}_{n,1}^{[s+1]}, M_n\right) \cong H_{q-1}\left(\mathbf{\Gamma}_{n,1}^{[s]}, H_1\left(\Sigma_{n,1}^{s}, R\right) \underset{R}{\otimes} M_n\right) \oplus H_q\left(\mathbf{\Gamma}_{n,1}^{[s]}, M_n\right). \tag{11}$$

Computation of $H_d\left(\mathbf{\Gamma}_{\infty,1}, H_1\left(\Sigma_{\infty,1}, \mathbb{Z}\right)^{\otimes m}\right)$. An application of Proposition 2.13 is to compute the stable homology groups $H_d\left(\mathbf{\Gamma}_{\infty,1}, H_1\left(\Sigma_{\infty,1}, \mathbb{Z}\right)^{\otimes m}\right)$ for all natural numbers m and d. First, let us introduce a suitable groupoid for our work, inspired by [28, Section 5.6]. We fix a unit disc with one marked point denoted by $\Sigma_{0,1}^{1}$ and a torus with one boundary component denoted by $\Sigma_{1,1}^{0}$. Recall that by the classification of surfaces, for all $g, s \in \mathbb{N}$, there is an homeomorphism $\Sigma_{g,1}^{s} \simeq \left(\underset{s}{\natural}\Sigma_{0,1}^{1}\right) \natural \left(\underset{g}{\natural}\Sigma_{1,1}^{0}\right)$.

Definition 2.14. Let $\mathfrak{M}_2$ be the skeleton of the groupoid defined by:

- Objects: the surfaces $\Sigma_{g,1}^{s}$ for all natural numbers g and s, with $I : [-1,1] \to \partial\Sigma_{g,1}^{s}$ a parametrized interval in the boundary and $p = 0 \in I$ a basepoint;
- Morphisms: $Aut_{\mathfrak{M}_2}\left(\Sigma_{g,1}^{s}\right) = \mathbf{\Gamma}_{g,1}^{s}$ for all natural numbers g and s.

Let $\mathfrak{M}_2^{g}$ be the full subgroupoid of $\mathfrak{M}_2$ on the objects $\{\Sigma_{n,1}\}_{n\in\mathbb{N}}$. As stated in the proof of [28, Proposition 5.18], the boundary connected sum $\natural$ induces a strict braided monoidal structure $(\mathfrak{M}_2^{g}, \natural, (\Sigma_{0,1}, I))$.

For all natural numbers g, we denote by $a_{\Sigma_{g,1}}$ the action of the mapping class group $\mathbf{\Gamma}_{g,1}$ on the fundamental group $\pi_1\left(\Sigma_{g,1}, p\right)$. We define a functor $\mathcal{A}_{\mathfrak{M}_2^{g}} : \mathfrak{M}_2^{g} \to \mathfrak{Gr}$ to be the fundamental groups $\pi_1\left(\Sigma_{1,1}, p\right)$ and $\pi_1\left(\Sigma_{0,1}, p\right)$ on the objects $\Sigma_{1,1}$ and $\Sigma_{0,1}$, and then inductively $\mathcal{A}_{\mathfrak{M}_2^{g}}\left(\Sigma_{n,1}\natural\Sigma_{n',1}\right) = \pi_1\left(\Sigma_{n,1}, p\right) * \pi_1\left(\Sigma_{n',1}, p\right)$ for all natural numbers n, and assigning the morphism $a_{\Sigma_{g,1}}(\varphi)$ for all $\varphi \in \mathbf{\Gamma}_{g,1}$. Recall that the group $\pi_1\left(\Sigma_{n,1}, p\right)$ is free of rank $2n$. By Van Kampen's theorem (see for example [20, Section 1.2]), the group $\mathcal{A}_{\mathfrak{M}_2^{g}}\left(\Sigma_{n,1}\natural\Sigma_{n',1}\right)$ is isomorphic to the fundamental group of the surface $\Sigma_{n,1}\natural\Sigma_{n',1}$: our assignment is thus consistent.

Hence, it follows from Lemma 2.4:

Proposition 2.15. *The functor $\mathcal{A}_{\mathfrak{M}_2^{g}} : \left(\mathfrak{M}_2^{g}, \natural, \Sigma_{0,1}^{0}\right) \to \left(\mathfrak{gr}, *, 0_{\mathfrak{Gr}}\right)$ is strict monoidal and extends to a functor $\pi_1(-,p) : \mathfrak{U}\mathfrak{M}_2^{g} \to \mathfrak{gr}$ by assigning for all natural numbers n and n':*

$$\pi_1(-,p)\left(\left[\Sigma_{n',1}, id_{\Sigma_{n'+n,1}}\right]\right) = \iota_{\pi_1\left(\Sigma_{n',1},p\right)} * id_{\pi_1\left(\Sigma_{n,1},p\right)}.$$

For all natural numbers n, since the free group $\pi_1\left(\Sigma_{n,1}, x\right)$ acts trivially on the homology group $H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)$, we have an isomorphism:

$$H_1\left(\pi_1\left(\Sigma_{n,1}, x\right), H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right) \cong H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes(m+1)}.$$

Also, the action of $\mathbf{\Gamma}_{n,2}$ on $H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}$ is induced by the one of $\mathbf{\Gamma}_{n,1}$ via the surjections $\omega_0 \circ \rho : \mathbf{\Gamma}_{n,2} \twoheadrightarrow \mathbf{\Gamma}_{n,1}^{1} \twoheadrightarrow \mathbf{\Gamma}_{n,1}$. It follows from Lemma 2.9 that the functor $H_1\left(\mathcal{A}_{\mathfrak{M}_2^{g}}, \mathbb{Z}\right)^{\otimes m}$ is very strong polynomial of degree m. Using the terminology of [4] and [8], $H_1\left(\mathcal{A}_{\mathfrak{M}_2^{g}}, \mathbb{Z}\right)^{\otimes m}$ is thus a coefficient system of degree m. Hence, it follows from the stability results of Boldsen [4] or Cohen and Madsen [8] that:

Theorem 2.16. *[4, Theorem 4.17][8, Theorem 0.4] Let m, n and q be natural numbers such that $2n \geq 3q + m$:*

$$H_q\left(\mathbf{\Gamma}_{n,2}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right) \cong H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right).$$

Then, we prove:

Theorem 2.17. *Let m, n and q be natural numbers such that $2n \geq 3q + m$. Then, there is an isomorphism:*

$$H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right) \cong \bigoplus_{\lfloor \frac{q-1}{2} \rfloor \geq k \geq 0} H_{q-(2k+1)}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m-1}\right).$$

Proof. For the purposes of this proof, we abbreviate $H_1(\Sigma_{n,1},\mathbb{Z})^{\otimes m}$ by $H^{(m)}$. The Lyndon-Hochschild-Serre spectral sequence with coefficients given by $H^{(m)}$ associated with the short exact sequence (10) has only two non-trivial rows. Hence, for all natural numbers $n \geq 1$, we obtain the following long exact sequence where we denote $\lambda_q = H_q\left(\rho, H^{(m)}\right)$:

$$\cdots \xrightarrow{d^2_{q+1,0}} H_{q-1}\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right) \longrightarrow H_q\left(\mathbf{\Gamma}_{n,2}, H^{(m)}\right) \xrightarrow{\lambda_q} H_q\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right) \xrightarrow{d^2_{q,0}} \cdots. \tag{12}$$

Recall from Corollary 2.3 that, denoting φ^{sp}_q a splitting of $\varphi_q : H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right) \to H_q\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right)$ (which exists by the splitting lemma for abelian groups), the isomorphism of Proposition 2.13 is defined by

$$\varphi^{sp}_q \oplus H_q\left(\omega_0, H^{(m)}\right) : H_q\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right) \cong H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right) \oplus H_q\left(\mathbf{\Gamma}_{n,1}, H^{(m)}\right).$$

Let us consider the projection $pr : H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right) \oplus H_q\left(\mathbf{\Gamma}_{n,1}, H^{(m)}\right) \twoheadrightarrow H_q\left(\mathbf{\Gamma}_{n,1}, H^{(m)}\right)$. Then, fixing a natural number n so that $2n \geq 3q + m$ and applying Theorem 2.16 (the isomorphism it gives being induced by $\omega_0 \circ \rho$), the composition

$$H_q\left(\omega_0 \circ \rho, H^{(m)}\right)^{-1} \circ pr \circ \left(\varphi^{sp}_q \oplus H_q\left(\omega_0, H^{(m)}\right)\right)$$

defines a splitting of λ_q. Therefore λ_q is split-injective and we obtain from the long exact sequence (12) the following isomorphism for $2n \geq 3q + m$:

$$H_q\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right) \cong H_q\left(\mathbf{\Gamma}_{n,2}, H^{(m)}\right) \oplus H_{q-2}\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right).$$

Now we again apply Proposition 2.13 to the homology groups $H_q\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right)$ and $H_{q-2}\left(\mathbf{\Gamma}^1_{n,1}, H^{(m)}\right)$. Then it follows from Theorem 2.16 that we have the following isomorphism for $2n \geq 3q + m$:

$$H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right) \oplus H_q\left(\mathbf{\Gamma}_{n,2}, H^{(m)}\right) \cong H_q\left(\mathbf{\Gamma}_{n,2}, H^{(m)}\right) \oplus H_{q-2}\left(\mathbf{\Gamma}_{n,1}, H^{(m)}\right) \oplus H_{q-3}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right).$$

Note that the summand $H_q\left(\mathbf{\Gamma}_{n,2}, H^{(m)}\right)$ on both side of this isomorphism is the image of the split-injective morphism λ_q. Hence the image through the differential $d^2_{q,0}$ gives the following isomorphism for $2n \geq 3q + m$:

$$H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right) \cong H_{q-2}\left(\mathbf{\Gamma}_{n,1}, H^{(m)}\right) \oplus H_{q-3}\left(\mathbf{\Gamma}_{n,1}, H^{(m+1)}\right).$$

The result then follows by induction on q. ☐

Remark 2.18. In [25, Theorem 1.B.], Kawazumi leads the analogous computation for cohomology. The method and techniques used in [25] are different from the ones presented here. Using the Universal Coefficient-type theorem for twisted coefficients (see for example [15, Théorème I.5.5.2]), Theorem 2.17 recovers the computation [25, Theorem 1.B.].

Computation of $H_d\left(\mathbf{\Gamma}^1_{\infty,1}, \mathbb{Z}\right)$. Another application of Proposition 2.13 is to compute the stable homology groups $H_d\left(\mathbf{\Gamma}^1_{\infty,1}, \mathbb{Z}\right)$ for all natural numbers d. Using Proposition 2.13 with constant module $\mathbb{Z}$ and Theorem 2.17 with $m = 1$, we prove:

Corollary 2.19. *Let n and q be natural numbers such that $2n \geq 3q$. Then, there is an isomorphism for all $m \geq 0$:*

$$\begin{aligned} H_q\left(\mathbf{\Gamma}^1_{n,1}, H_1(\Sigma_{n,1},\mathbb{Z})^{\otimes m}\right) \quad \cong \quad & \bigoplus_{\lfloor \frac{q-1}{2} \rfloor \geq k \geq 0} H_{q-(2k+1)}\left(\mathbf{\Gamma}_{n,1}, H_1(\Sigma_{n,1},\mathbb{Z})^{\otimes m-1}\right) \\ & \bigoplus_{\lfloor \frac{q-2}{2} \rfloor \geq k \geq 0} H_{q-(2k+2)}\left(\mathbf{\Gamma}_{n,1}, H_1(\Sigma_{n,1},\mathbb{Z})^{\otimes m}\right). \end{aligned}$$

In particular

$$H_q\left(\mathbf{\Gamma}^1_{n,1}, \mathbb{Z}\right) \cong \bigoplus_{\lfloor \frac{q}{2} \rfloor \geq k \geq 0} H_{q-2k}\left(\mathbf{\Gamma}_{n,1}, \mathbb{Z}\right).$$

Using other techniques (namely an equivalence of classifying spaces), Bödigheimer and Tillmann prove the more general result:

Theorem 2.20. *[3, Corollary 1.2] Let q and n be natural numbers such that $n \geq 2q$. For all natural numbers s*

$$H_q\left(\mathbf{\Gamma}^{[s]}_{n,1}, \mathbb{Z}\right) \cong \bigoplus_{k+l=q} \left(H_k\left(\mathbf{\Gamma}_{n,1}, \mathbb{Z}\right) \underset{\mathbb{Z}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times s}, \mathbb{Z}\right)\right)$$

where $\mathbb{C}P^\infty$ denotes the infinite dimensional complex projective space.

2.3.2 Automorphisms of free groups with boundaries

Let $\mathscr{G}_{n,k}$ denote the topological space consisting of a wedge of $n \in \mathbb{N}$ circles together with k distinguished circles joined by arcs to the basepoint. For $s \in \mathbb{N}$, let $\mathscr{G}^s_{n,k}$ be the space obtained from $\mathscr{G}_{n,k}$ by wedging $s-1$ edges at the basepoint. We denote by $\mathrm{Htpy}_*\left(\mathscr{G}^s_{n,k}; \partial\right)$ the space of homotopy equivalences of $\mathscr{G}_{n,k}$ that preserve the basepoint and restrict to the identity on each of the k distinguished circles and the s basepoints. Let $A^s_{n,k}$ be the group of path-components of $\mathrm{Htpy}_*\left(\mathscr{G}^s_{n,k}; \partial\right)$. For instance, for n a natural number and denoting by $\mathbf{F}_n$ the free group of rank n, then $A^1_{n,0}$ is isomorphic to the automorphism group of $\mathbf{F}_n$ (denoted by $Aut\left(\mathbf{F}_n\right)$) and $A^2_{n,0}$ is isomorphic to the holomorphs of the free group $\mathbf{F}_n$. We refer the reader to [21] and [23] for more details on these groups.

Remark 2.21. We denote by $M^s_{n,k}$ the connected sum $\left(\sharp_n\left(\mathbb{S}^1 \times \mathbb{S}^2\right)\right) \sharp \left(\sharp_k\left(\mathbb{S}^1 \times \mathbb{D}^2\right)\right) \sharp \left(\sharp_s\left(\mathbb{D}^3\right)\right)$ and by $\pi_0\mathrm{Diff}\left(M^s_{n,k}\right)$ the isotopy classes of diffeomorphisms of $M^s_{n,k}$ (a.k.a. the mapping class group of $M^s_{n,k}$). Recall from [21, Theorem 1.1] that there is a short exact sequence

$$1 \longrightarrow K^s_{n,k} \longrightarrow \pi_0\mathrm{Diff}\left(M^s_{n,k}\right) \longrightarrow A^s_{n,k} \longrightarrow 1\ ,$$

where $K^s_{n,k}$ is generated by Dehn twists along embedded 2 spheres (and is a product of at most $n+k+s$ copies of $\mathbb{Z}/2\mathbb{Z}$). Hence the automorphism of free groups with boundaries $A^s_{n,k}$ can be viewed as a quotient of the mapping class group of the 3-manifold $M^s_{n,k}$.

For $k, n \in \mathbb{N}$, we denote by $Aut_{n,k}$ the subgroup of $Aut\left(\mathbf{F}_{n+k}\right)$ of automorphisms that take each of the last k generators to a conjugate of itself. Denoting by $\mathrm{Htpy}_*\left(\mathscr{G}_{n,k}; [\partial]\right)$ the space of homotopy equivalences of $\mathscr{G}_{n,k}$ that preserve the basepoint and fixing the k distinguished circles up to a rotation, then $Aut_{n,k}$ is the group of path-components of $\mathrm{Htpy}_*\left(\mathscr{G}_{n,k}; [\partial]\right)$. Restricting the elements of $\mathrm{Htpy}_*\left(\mathscr{G}_{n,k}; [\partial]\right)$ to their rotations of the k distinguished circles defines a fibration $\mathrm{Htpy}_*\left(\mathscr{G}_{n,k}; \partial\right) \to \mathrm{Htpy}_*\left(\mathscr{G}_{n,k}; [\partial]\right) \to \left(\mathbb{S}^1\right)^k$. The homotopy long exact sequence associated with this fibration provides an exact sequence

$$\mathbb{Z}^k \longrightarrow A_{n,k} \longrightarrow Aut_{n,k} \longrightarrow 1$$

(actually, by [24, Section 2], the left hand map is injective and this is a short exact sequence), which gives a surjective map $A_{n,k} \twoheadrightarrow Aut_{n,k}$. For $k, n \in \mathbb{N}$, we denote by $a_{A_{n,k}}$ the composition $A^1_{n,k} \to A_{n,k} \twoheadrightarrow Aut_{n,k} \hookrightarrow Aut\left(\mathbf{F}_{n+k}\right)$ where the map $A^1_{n,k} \to A_{n,k}$ forgets the basepoint. We recall the following useful result:

Lemma 2.22. *[21] Let n, k and $s \geq 2$ be natural numbers. There is a split short exact sequence*

$$1 \longrightarrow \mathbf{F}_{n+k} \longrightarrow A^s_{n,k} \longrightarrow A^{s-1}_{n,k} \longrightarrow 1, \tag{13}$$

where the map $A^s_{n,k} \to A^{s-1}_{n,k}$ forgets the last basepoint and a fortiori $A^s_{n,k} \cong \left(\mathbf{F}_{n+k}\right)^{s-1} \rtimes A^1_{n,k}$ where $A^1_{n,k}$ acts diagonally on $\left(\mathbf{F}_{n+k}\right)^{s-1}$ via the map $a_{A_{n,k}} : A^1_{n,k} \to Aut\left(\mathbf{F}_{n+k}\right)$.

Let k and s be fixed natural numbers. Let $\mathfrak{A}_{s,k}$ be the groupoid with the spaces $\mathscr{G}^s_{n,k}$ as its objects and $A^s_{n,k}$ as automorphism groups for all natural numbers. For $s=1$ and $k=0$, $\mathfrak{A}_{1,0}$ is the maximal subgroupoid of the category $\mathfrak{gr}$ of finitely generated free groups and we denote by $i : \mathfrak{A}_{1,0} \to \mathfrak{gr}$ the inclusion functor. Moreover the coproduct $*$ induces a strict symmetric monoidal structure $\left(\mathfrak{A}_{1,0}, *, 0_{\mathfrak{Gr}}\right)$ by restriction and we thus define the symmetric monoidal category $(\mathfrak{UA}_{1,0},*,0_{\mathfrak{Gr}})$. The functor i lifts to the category $\mathfrak{UA}_{1,0}$ by sending a morphism $[\mathbf{F}_{n_2-n_1},g]:\mathbf{F}_{n_1}\to\mathbf{F}_{n_2}$ of $\mathfrak{UA}_{1,0}$ (where $g\in Aut(\mathbf{F}_{n_2})$) to the morphism $g\circ\left(\iota_{\mathbf{F}_{n_2-n_1}}*id_{\mathbf{F}_{n_1}}\right):\mathbf{F}_{n_1}\hookrightarrow\mathbf{F}_{n_2}$ of $\mathfrak{gr}$: since $f\circ\iota_{\mathbf{F}_n}=\iota_{\mathbf{F}_n}$ for all $f\in Aut(\mathbf{F}_n)$, the relation (2) is trivially satisfied and the result follows from Lemma 1.5. Then i is a faithful functor and $\mathfrak{UA}_{1,0}$ can be seen as a subcategory of $\mathfrak{gr}$.

Let k and $s\geq 1$ be natural numbers. Precomposing by the surjection $A^s_{n,k}\twoheadrightarrow A^{s-1}_{n,k}\twoheadrightarrow\cdots\twoheadrightarrow A^1_{n,k}$, the morphisms $\left\{a_{A_{n,k}}\right\}_{n\in\mathbb{N}}$ assemble to define a functor $\mathcal{A}_{\mathfrak{A}_{s,k}}:\mathfrak{A}_{s,k}\to\mathfrak{Gr}$ such that $\mathcal{A}_{\mathfrak{A}_{s,k}}(n)=\mathbf{F}_{n+k}$ for all natural numbers n. Furthermore, we recall the stable homology result for automorphism groups of free groups due to Galatius for constant coefficients and Djament and Vespa for twisted coefficients:

Theorem 2.23. *Let $q\geq 1$ be a natural number. Then:*

- *[13] for $n\geq 2q+1$, $H_q(Aut(\mathbf{F}_n),\mathbb{Q})=0$;*
- *[10, Théorème 1] for $F:\mathfrak{gr}\to\mathbf{Ab}$ a strong polynomial functor such that $F(0_{\mathfrak{Gr}})=0_{\mathfrak{Gr}}$, then*

$$\underset{n\in\mathbb{N}}{Colim}\left(H_q\left(Aut\left(\mathbf{F}_n\right),F(n)\right)\right)=0.$$

Hence, we can establish the main result of Section 2.3.2.

Theorem 2.24. *Let $s\geq 2$ and $q\geq 1$ be natural numbers.*

1. *Let $F:\mathfrak{gr}\to\mathbf{Ab}$ be a strong polynomial functor such that $F(0_{\mathfrak{Gr}})=0_{\mathfrak{Gr}}$. The action of $A^s_{n,0}$ on $F(n)$ is induced by the surjections $A^s_{n,0}\twoheadrightarrow A^{s-1}_{n,0}\twoheadrightarrow\cdots\twoheadrightarrow A^1_{n,0}$. Then*

$$\underset{n\in\mathbb{N}}{Colim}\left(H_q\left(A^s_{n,0},F(n)\right)\right)=0.$$

2. *For all natural numbers $n\geq 2q+2$ and $k\geq 0$, $H_q\left(A^s_{n,k},\mathbb{Q}\right)=0$.*

Proof. Since $i:\mathfrak{UA}_{1,0}\to\mathfrak{gr}$ is faithful and the monoidal structure of $\mathfrak{UA}_{1,0}$ is induced from the one of $\mathfrak{gr}$, i is strong monoidal. Recall from Remark 2.6 that, since the trivial group $0_{\mathfrak{Gr}}$ is a terminal object of $\mathfrak{gr}$, F is very strong polynomial. We then deduce from Proposition 2.7 that the functor $F\circ i:\mathfrak{UA}_{1,0}\to\mathbf{Ab}$ is very strong polynomial. It follows from Lemma 2.10 that $H_1(\mathbf{F}_-,\mathbb{Q})\underset{\mathbb{Q}}{\otimes}F(-):\mathfrak{UA}_{1,0}\overset{i}{\to}\mathfrak{gr}\to\mathbf{Ab}$ is a strong polynomial functor. Hence, the first result follows from Corollary 2.3 and Theorem 2.23.

In [22, Theorem 1.1], Hatcher and Wahl prove that the stabilization morphism $A^s_{n,k}\to A^s_{n,k+1}$ induces an isomorphism for the rational homology $H_q\left(A^s_{n,k},\mathbb{Q}\right)\overset{\sim}{\to}H_q\left(A^s_{n,k+1},\mathbb{Q}\right)$ if $n\geq 3q+3$. The second result thus follows from the previous statement. ☐

Remark 2.25. For $k=0$ and $s=2$, Theorem 2.24 recovers the results [23, Theorem 1.2 (b) and (c)] due to Jensen.

3 Twisted stable homologies for FI-modules

In this section, we present a general principle to compute the twisted stable homology for mapping class groups with non-trivial finite quotient groups. First, we give a general decomposition for the twisted stable homology using functor homology Section 3.1. Then, we can establish in Theorem 3.7 a general formula to compute the stable homology with twisted coefficients given by functors over categories associated with the aforementioned finite quotient groups in Section 3.2. This allows one to set explicit formulas for the stable homology with coefficients given by FI-modules for braid groups, mapping class groups of orientable surfaces and some particular right-angled Artin groups in Section 3.3. **Throughout Section 3, we fix $\mathbb{K}$ a field.**

3.1 General decomposition for the twisted stable homology using functor homology

In this first subsection, we prove a decomposition result for the stable homology with twisted coefficients for families of groups whose associated groupoid is braided strict monoidal and satisfies the assumptions of Theorem 1.12 (see Theorem 3.2). It extends a previous analogous result due Djament and Vespa in [9, Section 1 and 2] when the ambient monoidal structure is symmetric. It will be a key step to prove Theorem 3.7. We refer the reader to the papers [12, Section 2] and [9, Appendice A] for an introduction to homological algebra in functor categories and we assume that all the definitions, properties and results there are known.

Throughout Section 3.1, we consider $(\mathcal{G}, \natural, 0)$ a small braided strict monoidal groupoid, with objects indexed by the natural numbers. An object of $\mathcal{G}$ is thus denoted by $\underline{n}$, where n is its corresponding indexing natural number. We denote the automorphism group $Aut_{\mathcal{G}}(\underline{n})$ by G_n and assume that there are no morphisms between distinct objects. These groups define a family of groups $G_{-} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ such that $G_{-}(\gamma_n) : G_n \to G_{n+1}$ is the injective group morphism which sends $\varphi \in G_n$ to $\varphi \natural id_{\underline{1}}$.

We assume that $\mathcal{G}$ has no zero divisors, that $Aut_{\mathcal{G}}(0) = \{id_0\}$ and that it satisfies the properties (**C**) and (**I**) of Definition 1.11. By Theorem 1.12, the monoidal structure $\natural$ extends to Quillen's bracket construction and defines pre-braided homogeneous category $(\mathfrak{U}\mathcal{G}, \natural, 0)$. Also, the unit 0 is an initial object in $\mathfrak{U}\mathcal{G}$ and we recall that $\iota_{\underline{n}} = [\underline{n}, id_{\underline{n}}] : 0 \to \underline{n}$ denotes the unique morphism in $\mathfrak{U}\mathcal{G}$ from 0 to $\underline{n}$. Hence, we have canonical morphisms $id_{\underline{n}} \natural \iota_{\underline{n'-n}} : \underline{n} \to \underline{n'}$ in $\mathfrak{U}\mathcal{G}$, for all natural numbers n and n' such that $n' \geq n$.

We fix F an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod})$. Our goal is to compute the stable homology of the family of groups G_{-} with coefficients given by F. Namely, we are interested in the computation of

$$\underset{n \in (\mathbb{N}, \leq)}{\mathrm{Colim}} \left(H_* \left(G_n, F(\underline{n})\right)\right)$$

the colimit of the homology groups $H_*(G_n, F(\underline{n}))$ with respect to the morphisms

$$H_* \left(G_{-}(\gamma_n), F\left(id_{\underline{n}} \natural \iota_{\underline{1}}\right)\right) : H_*(G_n, F(\underline{n})) \to H_*(G_{n+1}, F(\underline{n+1}))$$

induced by the functoriality in two variables of group homology with respect to the morphisms $G_{-}(\gamma_n)$ and $F\left(id_{\underline{n}} \natural \iota_{\underline{1}}\right)$.

Notation 3.1. We denote by G_∞ the colimit of the groups G_n with respect to the morphisms $G_{-}(\gamma_n)$ and by $G_{-}(\gamma_\infty) : G_n \to G_\infty$ the associated canonical group morphism. Let F_∞ be the colimit of the G_n-modules $F(\underline{n})$ with respect to the morphisms $F\left(id_{\underline{n}} \natural \iota_{\underline{1}}\right)$. Let $H_*(G_\infty, F_\infty)$ be the homology group of the colimit G_∞ with coefficient in the colimit F_∞: it coincides with $\underset{n \in (\mathbb{N}, \leq)}{\mathrm{Colim}} \left(H_*(G(\underline{n}), F(\underline{n}))\right)$ (since group homology commutes with filtered colimits).

As categories with one object, the groups $\{G_n\}_{n \in \mathbb{N}}$ are subcategories of $\mathfrak{U}\mathcal{G}$: for each natural number n, we have a canonical faithful functor $\mathbf{i}_{G_n} : G_n \to \mathfrak{U}\mathcal{G}$. We denote by $\Pi : G_\infty \times \mathfrak{U}\mathcal{G} \to \mathfrak{U}\mathcal{G}$ the projection functor and by Π^* the precomposition by Π. Thanks to the functoriality of the homology of categories, we define for each natural number n a morphism

$$\Psi_{F,n} : H_*(G_n, F(n)) \to H_*(G_\infty \times \mathfrak{U}\mathcal{G}, \Pi^* F)$$

induced by restriction along the functor $(G_{-}(\gamma_\infty) \times \mathbf{i}_{G_n}) \circ \Delta_{G_n} : G_n \to G_\infty \times \mathfrak{U}\mathcal{G}$, where $\Delta_{G_n} : G_n \to G_n \times G_n$ is the diagonal functor. Then it formally follows from the previous definitions that for all natural numbers n:

$$H_*\left(G_{-}(\gamma_n), F\left(id_{\underline{n}} \natural \iota_{\underline{1}}\right)\right) \circ \Psi_{F,n+1} = \Psi_{F,n}.$$

A fortiori the morphisms $\{\Psi_{F,n}\}_{n \in \mathbb{N}}$ are natural with respect to $(\mathbb{N}, \leq)$. Hence, the colimit with respect to $(\mathbb{N}, \leq)$ defines a unique morphism:

$$\Psi_F : H_*(G_\infty, F_\infty) \to H_*(G_\infty \times \mathfrak{U}\mathcal{G}, \Pi^* F).$$

Let us state the main result of this section.

Theorem 3.2. *Let $\mathbb{K}$ be a field and $(\mathfrak{U}\mathcal{G}, \natural, 0)$ be a pre-braided homogeneous category as detailed before. For all functors $F : \mathfrak{U}\mathcal{G} \to \mathbb{K}\text{-}\mathfrak{Mod}$, the morphism Ψ_F is a $\mathbb{K}$-modules isomorphism. Moreover, Ψ_F decomposes as a natural isomorphism:*

$$H_*(G_\infty, F_\infty) \cong \bigoplus_{k+l=*} \left(H_k(G_\infty, \mathbb{K}) \underset{\mathbb{K}}{\otimes} H_l(\mathfrak{U}\mathcal{G}, F)\right).$$

Proof. Note that the morphism Ψ_F is a morphism of δ-functors commuting with filtered colimits. Recall that the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod})$ has enough projectives, provided by direct sums of the standard projective generators functors $P_{\underline{n}}^{\mathfrak{U}\mathcal{G}} = \mathbb{K}[Hom_{\mathfrak{U}\mathcal{G}}(\underline{n}, -)]$ for all natural numbers n. Therefore, we only have to show that Ψ_F is an isomorphism when $F = P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}$. Indeed, for an ordinary functor F, there is an epimorphism from a direct sum of standard projective generators to F and $\mathbb{K}\text{-}\mathfrak{Mod}$ is an abelian category: the result then follows from a straightforward recursion on the homological degree on the long exact sequence induced by this epimorphism.

We deduce from Lemma 1.10 that we have the following isomorphism of G_m-sets for all natural numbers $m \geq n$:

$$Hom_{\mathfrak{U}\mathcal{G}}(\underline{n}, \underline{m}) \cong G_m / G_{m-n}.$$

Hence $P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}(\underline{m}) \cong \mathbb{K}[G_m] \underset{\mathbb{K}[G_{m-n}]}{\otimes} \mathbb{K}$ as G_m-modules. Therefore, it follows from Shapiro's lemma that:

$$H_*\left(G_m, P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}(\underline{m})\right) \cong H_*(G_{m-n}, \mathbb{K}).$$

Taking the colimit with respect to m, we deduce the isomorphism $H_*\left(G_\infty, \left(P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}\right)_\infty\right) \cong H_*(G_\infty, \mathbb{K})$. Recall that $P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}$ is a projective object in $\mathfrak{U}\mathcal{G}$. Then, using the first Künneth spectral sequence for the product of two categories, the morphism $\Psi_{P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}}$ identifies with:

$$H_*\left(G_\infty, \left(P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}\right)_\infty\right) \cong H_*(G_\infty, \mathbb{K}) \cong H_*\left(G_\infty \times \mathfrak{U}\mathcal{G}, \Pi^*\left(P_{\underline{n}}^{\mathfrak{U}\mathcal{G}}\right)\right).$$

The second part of the statement follows applying Künneth formula for homology of categories. $\square$

3.2 Framework and first equivalence for stable homology

Throughout the remainder Section 3, we assume that the field $\mathbb{K}$ is of characteristic 0.

We consider three families of groups K_-, G_- and C_- which fit into the following short exact sequence in the category $\mathbf{Fct}((\mathbb{N}, \leq), \mathfrak{Gr})$:

$$0 \longrightarrow K_- \xrightarrow{k} G_- \xrightarrow{c} C_- \longrightarrow 0, \tag{14}$$

where $k : K_- \to G_-$ and $c : G_- \to C_-$ are natural transformations and 0 denotes the constant object of $\mathbf{Fct}((\mathbb{N}, \leq), \mathfrak{Gr})$ at $0_{\mathfrak{Gr}}$.

Let $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{C}$ denote the groupoids with objects indexed by natural numbers, with no morphisms between distinct objects, such that $Aut_{\mathcal{K}}(\underline{n}) = K_n$, $Aut_{\mathcal{G}}(\underline{n}) = G_n$ and $Aut_{\mathcal{C}}(\underline{n}) = C_n$. We assume that the groupoids $\mathcal{G}$ and $\mathcal{C}$ are endowed with braided strict monoidal structures $(\mathcal{G}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$ and $(\mathcal{C}, \natural_{\mathcal{C}}, 0_{\mathcal{C}})$, where $\natural_{\mathcal{G}}$ and $\natural_{\mathcal{C}}$ are defined by the addition on objects, such that:

- the morphisms $\{c_n\}_{n\in\mathbb{N}}$ induce a strict monoidal functor $\mathfrak{c} : \mathcal{G} \to \mathcal{C}$ defined by the identity on objects;
- $G_-(\gamma_n) = id_{\underline{1}} \natural_{\mathcal{G}} - : G_n \hookrightarrow G_{n+1}$ and $C_-(\gamma_n) = id_{\underline{1}} \natural_{\mathcal{C}} - : C_n \hookrightarrow C_{n+1}$ for all natural numbers n.

Recall that the associated Quillen's bracket construction $(\mathfrak{U}\mathcal{G}, \natural_{\mathcal{G}}, 0)$ and $(\mathfrak{U}\mathcal{C}, \natural_{\mathcal{C}}, 0)$ are pre-braided strict monoidal by Proposition 1.8. Let $\mathscr{O}'_{\mathcal{G}} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{G}$ and $\mathscr{O}'_{\mathcal{C}} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{C}$ be the faithful and essentially surjective functors assigning $\mathscr{O}'_{\mathcal{G}}(n) = \mathscr{O}'_{\mathcal{C}}(n) = \underline{n}$ and $\mathscr{O}'_{\mathcal{G}}(\gamma_n) = \mathscr{O}'_{\mathcal{C}}(\gamma_n) = \left[\underline{1}, id_{\underline{n+1}}\right]$ for all natural numbers n. Using the functors $\mathscr{O}'_{\mathcal{G}}$ and $\mathscr{O}'_{\mathcal{C}}$, the natural transformation $c : G_- \to C_-$ identifies the morphisms $\left[\underline{n' - n}, id_{\underline{n'}}\right]$ (with natural numbers $n' \geq n$) of $\mathfrak{U}\mathcal{G}$ and $\mathfrak{U}\mathcal{C}$. The criteria (1) and (2) of Lemma 1.5 being trivially checked, the functor $\mathfrak{c} : \mathcal{G} \to \mathcal{C}$ lifts to a functor $\mathfrak{U}\mathcal{G} \to \mathfrak{U}\mathcal{C}$, again denoted by $\mathfrak{c}$ (abusing the notation).

The short exact sequence (14) implies that the braided strict monoidal structure $(\mathcal{G}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$ induces a braided strict monoidal structure on $\mathcal{K}$, denoted by $(\mathcal{K}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$, such that:

$$K_-(\gamma_n) = id_1 \natural_{\mathcal{G}} - : K_n \hookrightarrow K_{n+1}$$

for all natural numbers n. As for the morphisms $\{c_n\}_{n\in\mathbb{N}}$, the morphisms $\{k_n\}_{n\in\mathbb{N}}$ induce a strict monoidal functor $\mathfrak{k} : \mathfrak{U}\mathcal{K} \to \mathfrak{U}\mathcal{G}$.

We fix F an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod})$. For all natural numbers n, we abuse the notation and write $F(n)$ for the restriction of $F(n)$ from G_n to K_n. Our aim is to compute the stable homology $H_*(G_\infty, F_\infty)$ of the family of groups G_- under the assumption that C_- is a family of finite groups. A first step is given by the following result:

Proposition 3.3. *We assume that the group C_n is finite for each natural number n. Then for all natural numbers q:*

$$H_q\left(G_n, F\left(\underline{n}\right)\right) \cong H_0\left(C_n, H_q\left(K_n, F\left(\underline{n}\right)\right)\right). \tag{15}$$

Proof. Applying the Lyndon-Hochschild-Serre spectral sequence for the short exact sequence (14), we obtain the following convergent first quadrant spectral sequence:

$$E^2_{pq} : H_p\left(C_n, H_q\left(K_n, F\left(\underline{n}\right)\right)\right) \Longrightarrow H_{p+q}\left(G_n, F\left(\underline{n}\right)\right). \tag{16}$$

Fixing n a natural number, we have for $p \neq 0$:

$$H_p\left(C_n, H_q\left(K_n, F\left(\underline{n}\right)\right)\right) = 0,$$

since C_n is a finite group. Hence, the second page of the spectral sequence (16) has non-zero terms only on the 0-th column and zero differentials. A fortiori, the convergence gives that $E^2 = E^\infty$ and this gives the desired result. □

Let us now focus on a key property for the homologies of the kernels $\{K_n\}_{n\in\mathbb{N}}$ which improves Proposition (3.3). Recall that, as K_n is a normal subgroup of G_n, the map $conj_n : G_n \to Aut_{\mathfrak{Gr}}\left(K_n\right)$ sending an element $g \in G_n$ to the left conjugation by g is a group morphism.

Lemma 3.4. *We define a functor $\tilde{K}_- : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ assigning $\tilde{K}_-\left(\underline{n}\right) = K_n$ for all natural numbers n and:*

1. *for all $g \in G_n$, $\tilde{K}_-\left(g\right) \in Aut_{\mathfrak{Gr}}\left(K_n\right)$ to be $conj_n\left(g\right) : k \mapsto gkg^{-1}$ for all $k \in K_n$,*
2. *$\tilde{K}_-\left(\left[1, id_{\underline{n+1}}\right]\right) = id_1 \natural_{\mathcal{G}} -$.*

Proof. It follows from the first assignment of Lemma 3.4 that we define a functor $\tilde{K}_- : \mathcal{G} \to \mathfrak{Gr}$. The relation (1) of Lemma 1.5 follows from the definition of the monoidal product $\natural_{\mathcal{G}}$. Let n and n' be natural numbers such that $n' \geq n$, let $g \in G_n$ and $g' \in G_{n'}$. We compute for all $k \in K_n$:

$$\begin{aligned}\left(\tilde{K}_-\left(g'\natural_{\mathcal{G}}g\right) \circ \tilde{K}_-\left(\left[\underline{n'}, id_{\underline{n'+n}}\right]\right)\right)(k) &= \left(g'\natural_{\mathcal{G}}g\right)\left(id_{n'}\natural_{\mathcal{G}}k\right)\left(g'\natural_{\mathcal{G}}g\right)^{-1} \\ &= id_{n'}\natural_{\mathcal{G}}\left(gkg^{-1}\right) \\ &= \left(\tilde{K}_-\left(\left[\underline{n'}, id_{\underline{n'+n}}\right]\right) \circ \tilde{K}_-\left(g\right)\right)(k).\end{aligned}$$

Hence, the relation (2) is satisfied a fortiori the result follows from Lemma 1.5. □

Lemma 3.4 is useful to prove the following key result.

Proposition 3.5. *For all natural numbers q, the homology groups $\left\{H_q\left(K_n, F\left(\underline{n}\right)\right)\right\}_{n\in\mathbb{N}}$ define a functor $H_q\left(K_-, F\left(-\right)\right) : \mathfrak{U}\mathcal{C} \to \mathbb{K}\text{-}\mathfrak{Mod}$.*

Proof. Let $\mathscr{P}$ be the category of pairs (G, M) where G is a group and M is a G-module for objects; for (G, M) and (G', M') objects of $\mathscr{P}$, a morphism from (G, M) to (G', M') is a pair (φ, α) where $\varphi \in Hom_{\mathfrak{Gr}}\left(G, G'\right)$ and $\alpha : M \to M'$ is a G-module morphism, where M' is endowed with a G-module structure via φ. Using the functor $F : \mathfrak{U}\mathcal{G} \to \mathbb{K}\text{-}\mathfrak{Mod}$, by Lemma 3.4 $\tilde{K}_-$ defines a functor $\left(\tilde{K}_-, F\left(-\right)\right) : \mathfrak{U}\mathcal{G} \to \mathscr{P}$. Recall from [5, Section 8] that group homology defines a covariant functor $H_* : \mathscr{P} \to \mathbb{K}\text{-}\mathfrak{Mod}$ for all $q \in \mathbb{N}$. Hence the composition with the functor $\left(\tilde{K}_-, F\left(-\right)\right) : \mathfrak{U}\mathcal{G} \to \mathscr{P}$ gives a functor:

$$H_q\left(K_-, F\left(-\right)\right) : \mathfrak{U}\mathcal{G} \to \mathbb{K}\text{-}\mathfrak{Mod}.$$

Moreover, since inner automorphisms act trivially in homology, we deduce that for all natural numbers n, the conjugation action of G_n on $\left(K_n, F\left(n\right)\right)$ induces an action of C_n on $H_*\left(K_n, F\left(n\right)\right)$. The monoidal structures $\left(\mathcal{G}, \natural_{\mathcal{G}}, 0_{\mathcal{G}}\right)$ and $\left(\mathcal{C}, \natural_{\mathcal{C}}, 0_{\mathcal{C}}\right)$ being compatible, we deduce that the functor $H_q\left(K_-, F\left(-\right)\right)$ factors through the category $\mathfrak{U}\mathcal{C}$ using the functor $\mathfrak{c} : \mathfrak{U}\mathcal{G} \to \mathfrak{U}\mathcal{C}$. □

Finally, we recall the following classical property for the homology of a category:

Proposition 3.6. *[12, Example 2.5] Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$ and let F be an object of $\mathbf{Fct}\left(\mathfrak{C}, R\text{-}\mathfrak{Mod}\right)$. Then, $H_0\left(\mathfrak{C}, F\right)$ is isomorphic to the colimit over $\mathfrak{C}$ of the functor $F : \mathfrak{C} \to R\text{-}\mathfrak{Mod}$.*

We thus deduce from Propositions 3.3 and 3.5:

Theorem 3.7. *Let K_-, G_- and C_- three families of groups fitting in the short exact sequence (14), such that the group C_n is finite for all natural numbers n and the groupoids $\mathcal{G}$ and $\mathcal{C}$ are endowed with the aforementioned braided strict monoidal structures $(\mathcal{G}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$ and $(\mathcal{C}, \natural_{\mathcal{C}}, 0_{\mathcal{C}})$. Then, for all natural numbers q:*

$$H_q\left(G_\infty, F_\infty\right) \cong \underset{l\in\mathfrak{U}\mathcal{C}}{Colim}\left(H_q\left(K_l, F\left(\underline{l}\right)\right)\right).$$

Moreover, if F factors through the category $\mathfrak{U}\mathcal{C}$ (in other words, $F : \mathfrak{U}\mathcal{G} \xrightarrow{\mathfrak{c}} \mathfrak{U}\mathcal{C} \to \mathbb{K}\text{-}\mathfrak{Mod}$), then:

$$H_q\left(G_\infty, F_\infty\right) \cong \underset{l\in\mathfrak{U}\mathcal{C}}{Colim}\left(H_q\left(K_l, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} F\left(\underline{l}\right)\right).$$

Proof. Applying Theorem 3.2 to Proposition 3.3, we obtain:

$$\underset{n\in\mathbb{N}}{Colim}\left(H_0\left(C_n, H_q\left(K_n, F\left(\underline{n}\right)\right)\right)\right) \cong \underset{n\in\mathbb{N}}{Colim}\left(H_0\left(C_n, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_0\left(\mathfrak{U}\mathcal{C}, H_q\left(K_-, F\right)\right)\right).$$

By Proposition 3.6, $H_0\left(\mathfrak{U}\mathcal{C}, H_q\left(K_-, F\right)\right) \cong \underset{l\in\mathfrak{U}\mathcal{C}}{Colim}\left(H_q\left(K_l, F\left(\underline{l}\right)\right)\right)$. Since $H_0\left(C_n, \mathbb{K}\right) \cong \mathbb{K}$, we deduce the first result. For the second result, requiring F to factor through $\mathfrak{U}\mathcal{C}$ is actually necessary to define the functor $H_q\left(K_-, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} F\left(-\right) : \mathfrak{U}\mathcal{C} \to \mathbb{K}\text{-}\mathfrak{Mod}$ using the pointwise tensor product of functors. Then the isomorphism follows from the universal coefficient theorem for homology. □

Remark 3.8. We can generalise Section 3.2 to the setting where $\mathbb{K}$ is a field of positive characteristic coprime to the cardinality of C_n for all n: the key point is that the homology group $H_p\left(C_n, H_q\left(K_n, F\left(\underline{n}\right)\right)\right) = 0$ has to vanish for $p \neq 0$ which is still true under this alternative assumption (since the multiplication by $|C_n|$ defines an isomorphism of $H_q\left(K_n, F\left(\underline{n}\right)\right)$).

3.3 Applications

We present now how to apply the general result of Theorem 3.7 for various families of groups. Beforehand, we fix some notations. We denote by $\mathfrak{S}_n$ the symmetric group on n elements and by $\mathfrak{S}_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ the family of groups defined by $\mathfrak{S}_-(n) = \mathfrak{S}_n$ and $\mathfrak{S}_-(\gamma_n) = id_1 \sqcup -$ for all natural numbers n.

Let Σ be the skeleton of the groupoid of finite sets and bijections. Note that $Obj\left(\Sigma\right) \cong \mathbb{N}$ and that the automorphism groups are the symmetric groups $\mathfrak{S}_n$. The disjoint union of finite sets $\sqcup$ induces a monoidal structure $(\Sigma, \sqcup, 0)$, the unit 0 being the empty set. This groupoid is symmetric monoidal, the symmetry being given by the canonical bijection $n_1 \sqcup n_2 \xrightarrow{\sim} n_2 \sqcup n_1$ for all natural numbers n_1 and n_2. The category $\mathfrak{U}\Sigma$ is equivalent to the category of finite sets and injections FI studied in [7].

3.3.1 Braid groups

We respectively denote by $\mathbf{B}_n$ the braid group on n strands and by $\mathbf{PB}_n$ the pure braid group on n strands. The braid groupoid $\boldsymbol{\beta}$ is the groupoid with objects the natural numbers $n \in \mathbb{N}$ and braid groups as automorphism groups. It is endowed with a strict braided monoidal product $\natural : \boldsymbol{\beta} \times \boldsymbol{\beta} \longrightarrow \boldsymbol{\beta}$, defined by the usual addition for the objects and laying two braids side by side for the morphisms. The object 0 is the unit of this monoidal product. The braiding of the strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ is defined for all natural numbers n and m by:

$$b^{\boldsymbol{\beta}}_{n,m} = \left(\sigma_m \circ \cdots \circ \sigma_2 \circ \sigma_1\right) \circ \cdots \circ \left(\sigma_{n+m-2} \circ \cdots \circ \sigma_n \circ \sigma_{n-1}\right) \circ \left(\sigma_{n+m-1} \circ \cdots \circ \sigma_{n+1} \circ \sigma_n\right)$$

where $\{\sigma_i\}_{i\in\{1,\ldots,n+m-1\}}$ denote the Artin generators of the braid group $\mathbf{B}_{n+m}$. We refer the reader to [26, Chapter XI, Section 4] for more details. By [28, Proposition 5.18], the category $\mathfrak{U}\boldsymbol{\beta}$ is pre-braided homogeneous.

The classical surjections $\left\{\mathbf{B}_n \overset{\mathfrak{p}_n}{\twoheadrightarrow} \mathfrak{S}_n\right\}_{n\in\mathbb{N}}$, sending each Artin generator $\sigma_i \in \mathbf{B}_n$ to the transposition $\tau_i \in \mathfrak{S}_n$ for all $i \in \{1, \ldots, n-1\}$ and for all natural numbers n, assemble to define a functor $\mathfrak{P} : \mathfrak{U}\boldsymbol{\beta} \to FI$. The functor

$\mathfrak{P}$ is strict monoidal with respect to the monoidal structures $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ and $(FI, \sqcup, 0)$. In addition, they define the following short exact sequence for all natural numbers n (see for example [2]):

$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \xrightarrow{\mathfrak{p}_n} \mathfrak{S}_n \longrightarrow 1 \ .$$

Let $\mathbf{PB}_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and $\mathbf{B}_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the families of groups defined by $\mathbf{PB}_-(n) = \mathbf{PB}_n$, $\mathbf{B}_-(n) = \mathbf{B}_n$ and $\mathbf{B}_-(\gamma_n) = \mathbf{PB}_-(\gamma_n) = id_1 \natural -$ for all natural numbers n. Therefore, by Theorem 3.7:

Proposition 3.9. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. For all natural numbers q, $H_q(\mathbf{B}_\infty, F_\infty) \cong \underset{n \in FI}{Colim}(H_q(\mathbf{PB}_n, F(n)))$, and if F factors through the category FI, then:*

$$H_q(\mathbf{B}_\infty, F_\infty) \cong \underset{n \in FI}{Colim}\left(H_q(\mathbf{PB}_n, \mathbb{K}) \underset{\mathbb{K}}{\otimes} F(n)\right).$$

The rational cohomology ring of the pure braid group on $n \in \mathbb{N}$ strands is computed by Arnol'd in [1]. Namely, $H^q(\mathbf{PB}_n, \mathbb{Q})$ is the graded exterior algebra generated by the classes $\omega_{i,j}$ for $i, j \in \{1, \ldots, n\}$ and $i < j$, subject to the relations $\omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0$. By the universal coefficient theorem for cohomology and as $H^q(\mathbf{PB}_n, \mathbb{K})$ is a finite-dimensional vector space, we deduce that $H_q(\mathbf{PB}_n, \mathbb{K}) \cong H^q(\mathbf{PB}_n, \mathbb{K})$.

Moreover, the FI-module structure of the homology groups $H_q(\mathbf{PB}_-, \mathbb{K})$ is well-known by [7, Example 5.1.A]: the conjugation action of the symmetric group $\mathfrak{S}_n$ on $H_q(\mathbf{PB}_n, \mathbb{K})$ translates into the permutation action of $\mathfrak{S}_n$ on the indices $i, j \in \{1, \ldots, n\}$ of the generators $\{\omega_{i,j}\}_{i,j \in \{1,\ldots,n\}}$. Hence, fixing some $F \in \mathbf{Fct}(FI, \mathbb{K}\text{-}\mathfrak{Mod})$, we have a complete description of the FI-module structure of the functor $H_q(\mathbf{PB}_-, \mathbb{K}) \underset{\mathbb{K}}{\otimes} F(-)$.

3.3.2 Mapping class group of orientable surfaces

We take the notations of Section 2.3.1. Recall that we introduced the groupoid $\mathfrak{M}_2$ associated with the surfaces $\Sigma^s_{n,1}$ for all natural numbers n and s. Let $\mathfrak{M}_2^=$ be the full subgroupoid of $\mathfrak{M}_2$ on the objects $\left\{\Sigma^n_{n,1}\right\}_{n \in \mathbb{N}}$. By [28, Proposition 5.18], the boundary connected sum $\natural$ induces a strict braided monoidal structure $\left(\mathfrak{M}_2^=, \natural, \left(\Sigma^0_{0,1}, I\right)\right)$ such that $\mathfrak{U}\mathfrak{M}_2^=$ is a pre-braided homogeneous category.

Recall that $\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma^n_{n,1}\right)$ denotes the space of diffeomorphisms of the surface $\Sigma^n_{n,1}$ which fix the boundary pointwise and fix the marked points. Let $\mathrm{Diff}^{\partial_0,\mathrm{permute}}\left(\Sigma^n_{n,1}\right)$ be the space of diffeomorphisms of the surface $\Sigma^n_{n,1}$ which fix the boundary pointwise and permute the marked points. The injections $\left\{\Gamma^{[n]}_{n,1} \overset{\mathfrak{i}_n}{\hookrightarrow} \Gamma^n_{n,1}\right\}_{n \in \mathbb{N}}$ induced by the inclusions

$$\mathrm{Diff}^{\partial_0,\mathrm{points}}\left(\Sigma^n_{n,1}\right) \hookrightarrow \mathrm{Diff}^{\partial_0,\mathrm{permute}}\left(\Sigma^n_{n,1}\right)$$

provide the following short exact sequence for all natural numbers n:

$$1 \longrightarrow \Gamma^{[n]}_{n,1} \xrightarrow{\mathfrak{i}_n} \Gamma^n_{n,1} \xrightarrow{\mathfrak{pm}_n} \mathfrak{S}_n \longrightarrow 1 \ .$$

The surjections $\{\mathfrak{pm}_n\}_{n \in \mathbb{N}}$ define a strict monoidal functor $\mathfrak{M}_2^= \to \Sigma$. Let $\Gamma^{[-]}_{-,1} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and $\Gamma^-_{-,1} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the families of groups defined by $\Gamma^{[-]}_{-,1}(n) = \Gamma^{[n]}_{n,1}$, $\Gamma^-_{-,1}(n) = \Gamma^n_{n,1}$ and $\Gamma^{[-]}_{-,1}(\gamma_n) = \Gamma^-_{-,1}(\gamma_n) = id_1 \natural -$ for all natural numbers n.

From Theorem 2.20, we deduce that for all natural numbers q such that $n \geq 2q$:

$$H_q\left(\Gamma^{[n]}_{n,1}, \mathbb{K}\right) \cong \bigoplus_{k+l=q}\left(H_k(\Gamma_{n,1}, \mathbb{K}) \underset{\mathbb{K}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n}, \mathbb{K}\right)\right).$$

The conjugation action of the symmetric group $\mathfrak{S}_n$ on $\Gamma^{[n]}_{n,1}$ is induced by the natural action of $\mathfrak{S}_n$ on $\Sigma^n_{n,1}$ given by permuting the marked points. Hence, according to the decomposition of the classifying space associated with the pure mapping class groups in [3, Theorem 1], the action of $\mathfrak{S}_n$ on $H_q\left(\Gamma^{[n]}_{n,1}, \mathbb{K}\right)$ corresponds to permuting the n factors $\mathbb{C}P^\infty$: the FI-module structure of the homology groups $H_q\left(\Gamma^{[n]}_{n,1}, \mathbb{K}\right)$ is thus well-understood using Künneth formula for $H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n},\mathbb{Z}\right)$. A fortiori the homology group $H_*\left(\Gamma_{n,1},\mathbb{K}\right)$ is a trivial $\mathfrak{S}_n$-module. Recall also from [27] that:

$$H_*\left(\Gamma_{\infty,1},\mathbb{K}\right)\cong\mathbb{K}\left[\kappa_1,\kappa_2,\ldots\right]$$

where each κ_i has degree $2i$.

By Theorem 3.7, we deduce that:

Proposition 3.10. *Let F be an object of $\mathbf{Fct}\left(\mathfrak{UM}_2^{=},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. For all natural numbers q,*

$$H_q\left(\Gamma^\infty_{\infty,1},F_\infty\right)\cong\underset{n\in FI}{Colim}\left(H_q\left(\Gamma^{[n]}_{n,1},F\left(n\right)\right)\right).$$

In particular, if F factors through the category FI, then:

$$H_q\left(\Gamma^\infty_{\infty,1},F_\infty\right)\cong\underset{n\in FI}{Colim}\left(\bigoplus_{k+l=q}\left(H_k\left(\Gamma_{n,1},\mathbb{K}\right)\underset{\mathbb{K}}{\otimes}H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n},\mathbb{K}\right)\right)\underset{\mathbb{K}}{\otimes}F\left(n\right)\right),$$

and a fortiori $H_{2k+1}\left(\Gamma^\infty_{\infty,1},F_\infty\right)=0$ for all natural numbers k.

Remark 3.11. The key point for the calculations of Section 3.3.2 to work is to consider a subgroupoid $\mathfrak{M}_2^{'}$ of $\mathfrak{M}_2$ so that the braided monoidal structure $\natural$ restricts to it (i.e $\left(\mathfrak{M}_2^{'},\natural,\left(\Sigma^0_{0,1},I\right)\right)$ is a braided monoidal groupoid) and so that the number of marked evolves linearly with respect to the genus: for instance we could consider the category with objects $\left\{\Sigma^{2n}_{n,1}\right\}_{n\in\mathbb{N}}$ and all the above results follow mutatis mutandis. However this does not change the result for the colimit.

3.3.3 Particular right-angled Artin groups

A right-angled Artin group (abbreviated RAAG) is a group with a finite set of generators $\left\{s_i\right\}_{1\leq i\leq k}$ with $k\in\mathbb{N}$ and relations $s_is_j=s_js_i$ for some $i,j\in\left\{1,\ldots,n\right\}$. For instance, the free group on k generators $\mathbf{F}_k$ is a RAAG. We refer to [31] or [14, Section 3] for more details on these groups.

By [14, Proposition 3.1], any RAAG admits a maximal decomposition as a direct product of RAAGs, unique up to isomorphism and permutation of the factors. A RAAG is said to be unfactorizable if its maximal decomposition is itself. We refer to [31] or [14, Section 3] for more details on these groups. Moreover, we have the following key property:

Lemma 3.12. *[14, Proposition 3.3] Let A be a fixed unfactorizable RAAG different from $\mathbb{Z}$. For all natural numbers n, we have the following split short exact sequence:*

$$1\longrightarrow Aut\left(A\right)^{\times n}\overset{i_n}{\longrightarrow}Aut\left(A^{\times n}\right)\overset{s_n}{\longrightarrow}\mathfrak{S}_n\longrightarrow 1.$$

Let $\mathcal{R}_A$ be the groupoid with the groups $A^{\times n}$ for all natural numbers n as its objects and $Aut\left(A^{\times n}\right)$ as automorphism groups. By [14, Sections 1 and 5], the direct product $\times$ induces a strict symmetric monoidal structure $\left(\mathcal{R}_A,\times,0_{\mathfrak{Gr}}\right)$ such that $\mathfrak{U}\mathcal{R}_A$ is a pre-braided homogeneous category. It is clear that the surjections $\left\{s_n\right\}_{n\in\mathbb{N}}$ define a strict monoidal functor $S:\mathcal{R}_A\to\Sigma$. Let $Aut\left(A^{\times-}\right):\left(\mathbb{N},\leq\right)\to\mathfrak{Gr}$ and $Aut\left(A\right)^{\times-}:\left(\mathbb{N},\leq\right)\to\mathfrak{Gr}$ be the families of groups defined by $Aut\left(A^{\times-}\right)\left(n\right)=Aut\left(A^{\times n}\right)$, $Aut\left(A\right)^{\times-}\left(n\right)=Aut\left(A\right)^{\times n}$ and $Aut\left(A^{\times-}\right)\left(\gamma_n\right)=Aut\left(A\right)^{\times-}\left(\gamma_n\right)=id_1\times-$ for all natural numbers n. By Theorem 3.7:

Proposition 3.13. *Let F be an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{R}_A,\mathbb{K}\text{-}\mathfrak{Mod}\right)$ and A be a fixed unfactorizable right-angled Artin group different from $\mathbb{Z}$. For all natural numbers q, $H_q\left(Aut\left(A^{\times\infty}\right),F_\infty\right)\cong\underset{n\in FI}{Colim}\left(H_q\left(Aut\left(A\right)^{\times n},F\left(n\right)\right)\right)$, and if F factors through the category FI, then:*

$$H_q\left(Aut\left(A^{\times\infty}\right),F_\infty\right)\cong\underset{n\in FI}{Colim}\left(H_q\left(Aut\left(A\right)^{\times n},\mathbb{K}\right)\underset{\mathbb{K}}{\otimes}F\left(n\right)\right).\tag{17}$$

Corollary 3.14. *Let A be a fixed unfactorizable right-angled Artin group different from $\mathbb{Z}$, such that there exists $N_A \in \mathbb{N}$ such that $H_q(Aut(A), \mathbb{K}) = 0$ for $1 \leq q \leq N_A$. Then, for all objects F of $\mathbf{Fct}(\mathfrak{UR}_A, \mathbb{K}\text{-}\mathfrak{Mod})$ factoring through the category FI:*

$$H_q\left(Aut\left(A^{\times\infty}\right), F_\infty\right) = 0,$$

for all natural numbers q such that $1 \leq q \leq N_A$.

Proof. It follows from Künneth Theorem that, for all natural numbers q such that $1 \leq q \leq N_A$, $H_q\left(Aut(A)^{\times n}, \mathbb{K}\right) = 0$. Then, the result follows from (17). □

Example 3.15. Let $\mathbf{F}_k$ be the free group on k generators. By [13, Corollary 1.2], for $k \geq 2q+1$ and $q \neq 0$, $H_q(Aut(\mathbf{F}_k), \mathbb{K}) = 0$. Let F be an object of $\mathbf{Fct}(\mathfrak{UR}_{\mathbf{F}_k}, \mathbb{K}\text{-}\mathfrak{Mod})$ factoring through the category FI. Then, for all natural numbers q and k such that $1 \leq q \leq \frac{k-1}{2}$:

$$H_q\left(Aut\left((\mathbf{F}_k)^{\times\infty}\right), F_\infty\right) = 0.$$

In particular, $H_q\left(Aut\left((\mathbf{F}_\infty)^{\times\infty}\right), F_\infty\right) = 0$ for all FI-module F.

References

[1] Vladimir I Arnold. The cohomology ring of the colored braid group. In *Vladimir I. Arnold-Collected Works*, pages 183–186. Springer, 1969.

[2] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[3] Carl-Friedrich Bödigheimer and Ulrike Tillmann. Stripping and splitting decorated mapping class groups. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 47–57. Birkhäuser, Basel, 2001.

[4] Søren K. Boldsen. Improved homological stability for the mapping class group with integral or twisted coefficients. *Math. Z.*, 270(1-2):297–329, 2012.

[5] Kenneth S Brown. *Cohomology of groups*, volume 87. Springer Science & Business Media, 2012.

[6] Jean Cerf. Topologie de certains espaces de plongements. *Bull. Soc. Math. France*, 89:227–380, 1961.

[7] Thomas Church, Jordan S. Ellenberg, and Benson Farb. FI-modules and stability for representations of symmetric groups. *Duke Math. J.*, 164(9):1833–1910, 2015.

[8] Ralph L. Cohen and Ib Madsen. Surfaces in a background space and the homology of mapping class groups. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 43–76. Amer. Math. Soc., Providence, RI, 2009.

[9] Aurélien Djament and Christine Vespa. Sur l'homologie des groupes orthogonaux et symplectiques à coefficients tordus. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(3):395–459, 2010.

[10] Aurélien Djament and Christine Vespa. Sur l'homologie des groupes d'automorphismes des groupes libres à coefficients polynomiaux. *Comment. Math. Helv.*, 90(1):33–58, 2015.

[11] Aurélien Djament and Christine Vespa. Foncteurs faiblement polynomiaux. *Int. Math. Res. Not. IMRN*, (2):321–391, 2019.

[12] Vincent Franjou and Teimuraz Pirashvili. Stable K-theory is bifunctor homology (after A. Scorichenko). In *Rational representations, the Steenrod algebra and functor homology*, volume 16 of *Panor. Synthèses*, pages 107–126. Soc. Math. France, Paris, 2003.

[13] Søren Galatius. Stable homology of automorphism groups of free groups. *Ann. of Math. (2)*, 173(2):705–768, 2011.

[14] Giovanni Gandini and Nathalie Wahl. Homological stability for automorphism groups of RAAGs. *Algebr. Geom. Topol.*, 16(4):2421–2441, 2016.

[15] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Actualités Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13. Hermann, Paris, 1958.

[16] André Gramain. Le type d'homotopie du groupe des difféomorphismes d'une surface compacte. In *Annales scientifiques de l'École Normale Supérieure*, volume 6, pages 53–66. Elsevier, 1973.

[17] Daniel Grayson. Higher algebraic K-theory: II (after Daniel Quillen). In *Algebraic K-theory*, pages 217–240. Lectures Notes in Math., Vol.551, Springer, Berlin, 1976.

[18] John Harer. The third homology group of the moduli space of curves. *Duke Math. J.*, 63(1):25–55, 1991.

[19] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Ann. of Math. (2)*, 121(2):215–249, 1985.

[20] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002.

[21] Allen Hatcher and Nathalie Wahl. Stabilization for the automorphisms of free groups with boundaries. *Geom. Topol.*, 9:1295–1336, 2005.

[22] Allen Hatcher and Nathalie Wahl. Stabilization for mapping class groups of 3-manifolds. *Duke Math. J.*, 155(2):205–269, 2010.

[23] Craig A. Jensen. Homology of holomorphs of free groups. *J. Algebra*, 271(1):281–294, 2004.

[24] Craig A. Jensen and Nathalie Wahl. Automorphisms of free groups with boundaries. *Algebr. Geom. Topol.*, 4:543–569, 2004.

[25] Nariya Kawazumi. On the stable cohomology algebra of extended mapping class groups for surfaces. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 383–400. Math. Soc. Japan, Tokyo, 2008.

[26] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[27] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.

[28] Oscar Randal-Williams and Nathalie Wahl. Homological stability for automorphism groups. *Adv. Math.*, 318:534–626, 2017.

[29] Arthur Soulié. The generalized Long-Moody functors. *arXiv:1709.04278.*, 2018.

[30] Arthur Soulié. The long–moody construction and polynomial functors. *Annales de l'Institut Fourier*, 69(4):1799–1856, 2019.

[31] Karen Vogtmann. $GL(n,\mathbb{Z})$, $Out(F_n)$ and everything in between: automorphism groups of RAAGs. In *Groups St Andrews 2013*, volume 422 of *London Math. Soc. Lecture Note Ser.*, pages 105–127. Cambridge Univ. Press, Cambridge, 2015.

[32] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

IRMA, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: artsou@hotmail.fr